

Optimal Ternary Constant-Composition Codes with Weight Four and Distance Six

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Abstract

The sizes of optimal constant-composition codes of weight three have been determined by Chee, Ge and Ling with four cases in doubt. Group divisible codes played an important role in their constructions. In this paper, we study the problem of constructing optimal ternary constant-composition codes with Hamming weight four and minimum distance six. The problem is solved with a small number of lengths undetermined. The previously known results are those with code length no greater than 10.

Key words and phrases: Constant-composition codes, group divisible codes, ternary codes

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1 Introduction

Constant-composition codes (CCCs) are a special type of constant-weight codes (CWCs) which are important in coding theory. The class of constant-composition codes includes the important permutation codes and has attracted recent interest due to their numerous applications, such as in determining the zero error decision feedback capacity of discrete memoryless channels [37], multiple-access communications [15], spherical codes for modulation [23], DNA codes [30, 32, 9], powerline communications [11, 13], frequency hopping [12], frequency permutation arrays [29], and coding for bandwidth-limited channels [14].

Systematic study began in late 1990's [3, 34, 5]. Today, various methods have been applied to the problem of determining the maximum size of a constant-composition code, such as

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Table 1: Values of $A_3(n, 6, [2, 2])$ and $A_3(n, 6, [3, 1])$ for $n \leq 10$

n	4	5	6	7	8	9	10
$A_3(n, 6, [2, 2])$	1	1	3	3	5	9	15
$A_3(n, 6, [3, 1])$	1	1	2	2	4	6	10

computer search methods [4], packing designs [18, 12, 19, 40, 42, 39, 41, 28], tournament designs [43], polynomials and nonlinear functions [17, 20, 12, 16, 21], difference triangle sets [6], PBD-closure methods [8, 7] and some other methods [35, 31].

In the paper of Svanström et al. [36], some methods for providing upper and lower bounds on the maximum size $A_3(n, d, \overline{w})$ of a ternary code with length n , minimum Hamming distance d , and constant composition \overline{w} were presented, and a table of exact values or bounds for $A_3(n, d, \overline{w})$ in the range $n \leq 10$ was also given there. Here we list the exact values of $A_3(n, 6, [2, 2])$ and $A_3(n, 6, [3, 1])$ for codes with length no greater than 10 in Table 1.

The sizes of optimal ternary CCCs with weight three have been determined completely by Chee, Ge and Ling in [7]. The sizes of optimal ternary CCCs with weight four and distance five have been determined completely by Gao and Ge in [24].

In this paper, we will concentrate our attention on ternary CCCs with weight four and distance six. We shall use group divisible codes as the main tools, which were first introduced by Chee et al. in [7] and have been shown to be useful in recursive constructions of CWCs and CCCs. The article is organized as follows. Section II provides some basic definitions and results on combinatorial designs and coding theory. In Sections III and IV, we focus on the determination for the exact values of $A_3(n, 6, [2, 2])$ and $A_3(n, 6, [3, 1])$ respectively. A brief conclusion is presented in Section V.

2 Preliminaries

2.1 Definitions and Notations

The set of integers $\{i, i+1, \dots, j\}$ is denoted by $[i, j]$. When $i = 0$ and $j = q-1$, the set is also denoted by I_q . The ring $\mathbb{Z}/q\mathbb{Z}$ is denoted by \mathbb{Z}_q . The notation $\{\cdot\}$ is used for multisets.

All sets considered in this paper are finite if not obviously infinite. If X and R are finite sets, R^X denotes the set of vectors of length $|X|$, where each component of a vector $u \in R^X$ has value in R and is indexed by an element of X , that is, $u = (u_x)_{x \in X}$, and $u_x \in R$ for each $x \in X$. A q -ary code of length n is a set $\mathcal{C} \subseteq \mathbb{Z}_q^X$ for some X with size n . The elements of \mathcal{C} are called *codewords*. The *support* of a vector $u \in \mathbb{Z}_q^X$, denoted $\text{supp}(u)$, is the set $\{x \in X : u_x \neq 0\}$. The *Hamming norm* or the *Hamming weight* of a vector $u \in \mathbb{Z}_q^X$ is defined as $\|u\| = |\text{supp}(u)|$. The distance induced by this norm is called the *Hamming distance*, denoted d_H , so that $d_H(u, v) = \|u - v\|$, for $u, v \in \mathbb{Z}_q^X$. A code \mathcal{C} is said to have *minimum distance* d if $d_H(u, v) \geq d$ for all distinct $u, v \in \mathcal{C}$. The *composition* of a vector

$u \in \mathbb{Z}_q^X$ is the tuple $\bar{w} = [w_1, \dots, w_{q-1}]$, where $w_j = |\{x \in X : u_x = j\}|$. A code \mathcal{C} is said to have *constant weight* w if every codeword in \mathcal{C} has weight w , and is said to have *constant composition* \bar{w} if every codeword in \mathcal{C} has composition \bar{w} . Hence, every constant-composition code is a constant-weight code. We refer to a q -ary code of length n , distance d , and constant weight w as an $(n, d, w)_q$ -code. If in addition, the code has constant composition \bar{w} , then it is referred to as an $(n, d, \bar{w})_q$ -code. The maximum size of an $(n, d, w)_q$ -code is denoted as $A_q(n, d, w)$ and that of an $(n, d, \bar{w})_q$ -code is denoted as $A_q(n, d, \bar{w})$. Any $(n, d, w)_q$ -code or $(n, d, \bar{w})_q$ -code achieving the maximum size is called *optimal*.

The following operations do not affect distance and weight properties of an $(n, d, \bar{w})_q$ -code:

- (i) reordering the components of \bar{w} , and
- (ii) deleting zero components of \bar{w} .

Consequently, throughout this paper, we restrict our attention to those compositions $\bar{w} = [w_1, \dots, w_{q-1}]$, where $w_1 \geq \dots \geq w_{q-1} \geq 1$.

Suppose $u \in \mathbb{Z}_q^X$ is a codeword of an $(n, d, \bar{w})_q$ -code, where $\bar{w} = [w_1, \dots, w_{q-1}]$. Let $w = \sum_{i=1}^{q-1} w_i$. We can represent u equivalently as a w -tuple $\langle a_1, a_2, \dots, a_w \rangle \in X^w$, where

$$\begin{aligned} u_{a_1} &= \dots = u_{a_{w_1}} = 1, \\ u_{a_{w_1+1}} &= \dots = u_{a_{w_1+w_2}} = 2, \\ &\vdots \\ u_{a_{\sum_{i=1}^{q-2} w_i+1}} &= \dots = u_w = q-1. \end{aligned}$$

Throughout this paper, we shall often represent codewords of constant-composition codes in this form. This has the advantage of being more succinct and more flexible in manipulation.

2.2 Upper Bounds

For constant-composition codes, we have

Lemma 2.1 (Chee et al. [7, 6])

$$A_q(n, d, [w_1, \dots, w_{q-1}]) = \begin{cases} \binom{n}{\sum_{i=1}^{q-1} w_i} \binom{\sum_{i=1}^{q-1} w_i}{w_1, \dots, w_{q-1}}, & \text{if } d \leq 2 \\ \lfloor n/w_1 \rfloor, & \text{if } d = 2 \sum_{i=1}^{q-1} w_i - 1 \text{ and } n \text{ is sufficiently large} \\ \left\lfloor \frac{n}{\sum_{i=1}^{q-1} w_i} \right\rfloor, & \text{if } d = 2 \sum_{i=1}^{q-1} w_i \\ 1, & \text{if } d \geq 2 \sum_{i=1}^{q-1} w_i + 1. \end{cases}$$

The following Johnson-type bound has been proven for constant-composition codes.

Theorem 2.2 (Svanström et al. [36])

$$A_q(n, d, [w_1, \dots, w_{q-1}]) \leq \frac{n}{w_1} A_q(n-1, d, [w_1-1, \dots, w_{q-1}]).$$

As a consequence of Lemma 2.1 and Theorem 2.2, we have the following result.

Corollary 2.3

$$A_3(n, 6, [2, 2]) \leq \left\lfloor \frac{n}{2} \left\lfloor \frac{n-1}{3} \right\rfloor \right\rfloor,$$

$$A_3(n, 6, [3, 1]) \leq \left\lfloor \frac{n}{3} \left\lfloor \frac{n-1}{3} \right\rfloor \right\rfloor.$$

For most cases, we will show that the above Johnson-type bound is tight. However, for the cases $n \equiv 4, 5, 7 \pmod{9}$ and $\bar{w} = [3, 1]$, other arguments can give better bounds.

Lemma 2.4

$$A_3(9t+4, 6, [3, 1]) \leq 9t^2 + 6t + 1 + \lfloor \frac{t}{4} \rfloor,$$

$$A_3(9t+5, 6, [3, 1]) \leq 9t^2 + 7t + 1 + \lfloor \frac{t+1}{4} \rfloor,$$

$$A_3(9t+7, 6, [3, 1]) \leq 9t^2 + 11t + 3 + \lfloor \frac{t+1}{2} \rfloor.$$

Proof: Let \mathcal{C} be a code of length n with composition $[3, 1]$, minimum distance six and M codewords. Let \mathcal{C}_i^1 denote the set of codewords with 1 in position i and let \mathcal{C}_i^2 denote the set of codewords with 2 in position i . Let x_i, y_i be the sizes of \mathcal{C}_i^1 and \mathcal{C}_i^2 respectively. Count the number of nonzero symbols in the code in two ways to get $4M = \sum_i (x_i + y_i)$. We want to bound the value of $x_i + y_i$. Consider a fixed position i of \mathcal{C} . Since the minimum distance of \mathcal{C} is six, the remaining nonzero symbols of the codewords in \mathcal{C}_i^1 should lie in different positions. So we have $3x_i \leq n-1$. Now, we consider a codeword in \mathcal{C}_i^2 . Such a codeword cannot have its 1s in any of the $2x_i$ positions where a codeword of \mathcal{C}_i^1 has a 1, as this would give a minimum distance smaller than six. Furthermore, the 1s of two codewords in \mathcal{C}_i^2 cannot overlap, so we have $2x_i + 3y_i \leq n-1$.

When $n = 9t+4$, it follows from $2x_i + 3y_i \leq 9t+3$ that $x_i + y_i \leq 3t+1 + \lfloor \frac{x_i}{3} \rfloor$. Noting that $3x_i \leq 9t+3$, we can get $x_i + y_i \leq 4t+1$. Thus $M \leq \lfloor \frac{(9t+4)(4t+1)}{4} \rfloor$. We then obtain the first inequation. The other inequations can be obtained by similar arguments. \square

In the rest of this paper, we use the notation $U(n, 6, [2, 2]) = \lfloor \frac{n}{2} \lfloor \frac{n-1}{3} \rfloor \rfloor$ as the upper bound for the maximum size of an $(n, 6, [2, 2])_3$ -code. For the composition $[3, 1]$, when

$n \equiv 0, 1, 2, 3, 6, 8 \pmod{9}$, we denote $U(n, 6, [3, 1]) = \lfloor \frac{n}{3} \lfloor \frac{n-1}{3} \rfloor \rfloor$; for $n \equiv 4, 5, 7 \pmod{9}$, write $n = 9t + i$ with $i = 4, 5$ or 7 and denote

$$U(9t + 4, 6, [3, 1]) = 9t^2 + 6t + 1 + \lfloor \frac{t}{4} \rfloor,$$

$$U(9t + 5, 6, [3, 1]) = 9t^2 + 7t + 1 + \lfloor \frac{t+1}{4} \rfloor,$$

$$U(9t + 7, 6, [3, 1]) = 9t^2 + 11t + 3 + \lfloor \frac{t+1}{2} \rfloor.$$

2.3 Designs

Our recursive construction is based on some combinatorial structures in design theory. The most important tools are pairwise balanced designs (PBDs) and group divisible designs (GDDs).

Let K be a subset of positive integers and let λ be a positive integer. A *pairwise balanced design* $((v, K, \lambda)$ -PBD or (K, λ) -PBD of order v) is a pair (X, \mathcal{B}) , where X is a finite set (*the point set*) of cardinality v and \mathcal{B} is a family of subsets (*blocks*) of X that satisfy (1) if $B \in \mathcal{B}$, then $|B| \in K$ and (2) every pair of distinct elements of X occurs in exactly λ blocks of \mathcal{B} . The integer λ is the index of the PBD.

Theorem 2.5 (Abel et al. [1]) *For any integer $v \geq 10$, a $(v, \{4, 5, 6, 7, 8, 9\}, 1)$ -PBD exists with exceptions $v \in \{10, 11, 12, 14, 15, 18, 19, 23\}$.*

Theorem 2.6 (Abel et al. [1]) *For any integer $v \geq 10$, a $(v, \{5, 6, 7, 8, 9\}, 1)$ -PBD exists with exceptions $v \in [10, 20] \cup [22, 24] \cup [27, 29] \cup [32, 34]$.*

A *group divisible design* (GDD) is a triple $(X, \mathcal{G}, \mathcal{B})$ where X is a set of points, \mathcal{G} is a partition of X into *groups*, and \mathcal{B} is a collection of subsets of X called *blocks* such that any pair of distinct points from X occurs either in some group or in exactly one block, but not both. A K -GDD of type $g_1^{u_1} g_2^{u_2} \dots g_s^{u_s}$ is a GDD in which every block has size from the set K and in which there are u_i groups of size $g_i, i = 1, 2, \dots, s$. When $K = \{k\}$, we simply write k for K . A *parallel class* or *resolution class* is a collection of blocks that partitions the point set of the design. A GDD is *resolvable* if the blocks of the design can be partitioned into parallel classes. A resolvable GDD is denoted by RGDD.

A k -GDD of type m^k is also called a *transversal design* and denoted by $TD(k, m)$.

Theorem 2.7 (Abel et al. [2]) *Let m be a positive integer. Then:*

- i) a $TD(4, m)$ exists if $m \notin \{2, 6\}$;
- ii) a $TD(5, m)$ exists if $m \notin \{2, 3, 6, 10\}$;

- (iii) a $TD(6, m)$ exists if $m \notin \{2, 3, 4, 6, 10, 14, 18, 22\}$;
- (iv) a $TD(7, m)$ exists if $m \notin \{2, 3, 4, 5, 6, 10, 14, 15, 18, 20, 22, 26, 30, 34, 38, 46, 60\}$;
- v) a $TD(8, m)$ exists if $m \notin \{2, 3, 4, 5, 6, 10, 12, 14, 15, 18, 20, 21, 22, 26, 28, 30, 33, 34, 35, 38, 39, 42, 44, 46, 51, 52, 54, 58, 60, 62, 66, 68, 74\}$;
- vi) a $TD(m+1, m)$ exists if m is a prime power.

A *double group divisible design* (DGDD) is a quadruple $(X, \mathcal{H}, \mathcal{G}, \mathcal{B})$ where X is a set of points, \mathcal{H} and \mathcal{G} are partitions of X (into *holes* and *groups*, respectively) and \mathcal{B} is a collection of subsets of X (*blocks*) such that

- (i) for each block $B \in \mathcal{B}$ and each hole $H \in \mathcal{H}$, $|B \cap H| \leq 1$,
- (ii) any pair of distinct points from X which are not in the same hole occurs either in some group or in exactly one block, but not both.

A K -DGDD of type $(g_1, h_1^v)^{u_1} (g_2, h_2^v)^{u_2} \dots (g_s, h_s^v)^{u_s}$ is a double group divisible design in which every block has size from the set K and in which there are u_i groups of size g_i , each of which intersects each of the v holes in h_i points. A *modified group divisible design* K -MGDD of type g^u is a K -DGDD of type $(g, 1^g)^u$.

2.4 Group Divisible Codes

Given $u \in \mathbb{Z}_q^X$ and $Y \subseteq X$, the *restriction of u to Y* , written $u|_Y$, is the vector $v \in \mathbb{Z}_q^Y$ such that

$$v_x = \begin{cases} u_x, & \text{if } x \in Y \\ 0, & \text{if } x \in X \setminus Y. \end{cases}$$

A *group divisible code* (GDC) of distance d is a triple $(X, \mathcal{G}, \mathcal{C})$, where $\mathcal{G} = \{G_1, \dots, G_t\}$ is a partition of X with cardinality $|X| = n$ and $\mathcal{C} \subseteq \mathbb{Z}_q^X$ is a q -ary code of length n , such that $d_H(u, v) \geq d$ for each distinct $u, v \in \mathcal{C}$, and $\|u|_{G_i}\| \leq 1$ for each $u \in \mathcal{C}$, $1 \leq i \leq t$. Elements of \mathcal{G} are called groups. We denote a GDC $(X, \mathcal{G}, \mathcal{C})$ of distance d as w -GDC(d) if \mathcal{C} is of constant weight w . If we want to emphasize the composition of the codewords, we denote the GDC as \overline{w} -GDC(d) when every $u \in \mathcal{C}$ has composition \overline{w} . The type of a GDC $(X, \mathcal{G}, \mathcal{C})$ is the multiset $\{|G| : G \in \mathcal{G}\}$. As in the case of GDDs, the exponential notation is used to describe the type of a GDC. The size of a GDC $(X, \mathcal{G}, \mathcal{C})$ is $|\mathcal{C}|$. Note that an $(n, d, \overline{w})_q$ -code with size s is equivalent to a \overline{w} -GDC(d) of type 1^n with size s .

Constant-composition codes of larger orders can often be obtained from GDCs via the following two constructions.

Construction 2.8 (Filling in Groups, see [7]) *Let $d \leq 2(w-1)$. Suppose there exists a w -GDC(d) $(X, \mathcal{G}, \mathcal{C})$ of type $g_1^{t_1} \dots g_s^{t_s}$ with size a . Suppose further that for each i , $1 \leq i \leq s$,*

there exists a $(g_i, d, w)_q$ -code \mathcal{C}_i with size b_i , then there exists a $(\sum_{i=1}^s t_i g_i, d, w)_q$ -code \mathcal{C}' with size $a + \sum_{i=1}^s t_i b_i$. In particular, if \mathcal{C} and \mathcal{C}_i , $1 \leq i \leq s$, are of constant composition \overline{w} , then \mathcal{C}' is also of constant composition \overline{w} .

Construction 2.9 (Adjoining y Points, see [7]) Let $d \leq 2(w-1)$ and $y \in \mathbb{Z}_{\geq 0}$. Suppose there exists a (master) w -GDC(d) of type $g_1^{t_1} \cdots g_s^{t_s}$ with size a , and suppose the following (ingredients) also exist:

- (i) a $(g_1 + y, d, w)_q$ -code with size b ,
- (ii) a w -GDC(d) of type $1^{g_i} y^1$ with size c_i for each $2 \leq i \leq s$,
- (iii) a w -GDC(d) of type $1^{g_1} y^1$ with size c_1 if $t_1 \geq 2$.

Then, there exists a $(y + \sum_{i=1}^s t_i g_i, d, w)_q$ -code with size $a + b + (t_1 - 1)c_1 + \sum_{i=2}^s t_i c_i$. Furthermore, if the master and ingredient codes are of constant composition, then so is the resulting code.

The following two constructions are useful for generating GDCs of larger orders from smaller ones.

Construction 2.10 (Fundamental Construction [7]) Let $d \leq 2(w-1)$, $\mathcal{D} = (X, \mathcal{G}, \mathcal{A})$ be a (master) GDD, and $\omega : X \rightarrow \mathbb{Z}_{\geq 0}$ be a weight function. Suppose that for each $A \in \mathcal{A}$, there exists an (ingredient) w -GDC(d) of type $\{\omega(a) : a \in A\}$. Then there exists a w -GDC(d) \mathcal{D}^* of type $\{\sum_{x \in G} \omega(x) : G \in \mathcal{G}\}$. Furthermore, if the ingredient GDCs are of constant composition \overline{w} , then \mathcal{D}^* is also of constant composition \overline{w} .

Construction 2.11 (Inflation Construction) Let $d \leq 2(w-1)$. Suppose there exists a w -GDC(d) of type $g_1^{t_1} \cdots g_s^{t_s}$ with size a . Suppose further that there exists a TD(w, m), then there exists a w -GDC(d) of type $(mg_1)^{t_1} \cdots (mg_s)^{t_s}$ with size am^2 . If the original GDC is of constant composition \overline{w} , then so is the derived GDC.

3 Determining the Value of $A_3(n, 6, [2, 2])$

In this section, we focus on the determination for the exact values of $A_3(n, 6, [2, 2])$ for all positive integers n . We first construct some $[2, 2]$ -GDC(6)s to obtain the optimal $(n, 6, [2, 2])_3$ -codes.

3.1 Skew Room Frame Construction

If $\{S_1, \dots, S_n\}$ is a partition of a set S , an $\{S_1, \dots, S_n\}$ -Room frame is an $|S| \times |S|$ array, F , indexed by S , satisfying:

1. every cell of F either is empty or contains an unordered pair of symbols of S ,
2. the subarrays $S_i \times S_i$ are empty, for $1 \leq i \leq n$ (these subarrays are *holes*),
3. each symbol $x \notin S_i$ occurs once in row (or column) s for any $s \in S_i$, and
4. the pairs occurring in F are those $\{s, t\}$, where $(s, t) \in (S \times S) \setminus \bigcup_{i=1}^n (S_i \times S_i)$.

The *type* of an $\{S_1, \dots, S_n\}$ -Room frame F will be the multiset $\{ |S_1|, \dots, |S_n| \}$. We will say that F has type $t_1^{u_1} \dots t_k^{u_k}$ provided there are u_j S_i 's of cardinality t_j , for $1 \leq j \leq k$. A Room frame is *skew* if cell (i, j) is filled implies that cell (j, i) is empty. A Room frame of type 1^n is called a *Room square*.

Lemma 3.1 ([10], [45]) *The necessary conditions for the existence of a skew Room frame of type t^u , namely, $u \geq 4$ and $t(u-1)$ is even, are also sufficient except for $(t, u) \in \{(1, 5), (2, 4)\}$ and with possible exceptions:*

- (i) $u = 4$ and $t \equiv 2 \pmod{4}$,
- (ii) $u = 5$ and $t \in \{17, 19, 23, 29, 31\}$.

Proposition 3.2 *If there exists a skew Room frame of type t^u , then there exists a $[2, 2]$ -GDC(6) of type $(6t)^u$ with size $6t^2u(u-1)$.*

Proof: Let F be a given skew Room frame of type t^u . We construct a $[2, 2]$ -GDC(6) of type $(6t)^u$ on group set $\{(i+k, j) : 0 \leq i \leq t-1, j \in \mathbb{Z}_6\} : k = 0, t, \dots, t(u-1)\}$. The code contains all the codewords $\langle (a, j), (b, j), (c, 1+j), (r, 4+j) \rangle, \langle (c, 4+j), (r, 1+j), (a, j), (b, j) \rangle$, where $j \in \mathbb{Z}_6$ and the pair $\{a, b\}$ is contained in column c and row r of F . It is easy to check that this code is of distance 6 and composition $[2, 2]$. \square

Combining Lemma 3.1 and Proposition 3.2, we have the following result.

Lemma 3.3 *Let $u \geq 4$ and $t(u-1)$ be even. Then there exists a $[2, 2]$ -GDC(6) of type $(6t)^u$ with size $6t^2u(u-1)$, except possibly for:*

- (i) $u = 4$ and $t \equiv 2 \pmod{4}$,
- (ii) $u = 5$ and $t \in \{1, 17, 19, 23, 29, 31\}$.

3.2 Difference Matrix Construction

Let G be an abelian group of order g . A *difference matrix* based on G , denoted $(g, k; 1)$ -DM, is a $k \times g$ matrix $M = [m_{i,j}]$, $m_{i,j}$ in G , such that for each $1 \leq r < s \leq k$, the differences $m_{r,j} - m_{s,j}$, $1 \leq j \leq g$, comprise all the elements of G .

Theorem 3.4 ([25]) *A $(g, 4; 1)$ -DM exists if and only if $g \geq 4$ and $g \not\equiv 2 \pmod{4}$.*

Proposition 3.5 *If there exists a $(g, 4; 1)$ -DM over \mathbb{Z}_g , then there exists a $[2, 2]$ -GDC(6) of type g^4 with size $2g^2$.*

Proof: Let $M = [m_{i,j}]$ be a given $(g, 4; 1)$ -DM over \mathbb{Z}_g . We construct a $[2, 2]$ -GDC(6) of type g^4 on group set $\{\{(i, j) : j \in \mathbb{Z}_g\} : 0 \leq i \leq 3\}$. The code contains all the codewords $\langle (0, m_{1,j} + k), (1, m_{2,j} + k), (2, m_{3,j} + k), (3, m_{4,j} + k) \rangle, \langle (2, m_{3,j} + k), (3, m_{4,j} + k), (0, m_{1,j} + 1 + k), (1, m_{2,j} + 1 + k) \rangle$, where $j, k \in \mathbb{Z}_g$. It is easy to check that this code is of distance 6 and composition $[2, 2]$. \square

Combining Theorem 3.4 and Proposition 3.5, we have the following result.

Lemma 3.6 *There exists a $[2, 2]$ -GDC(6) of type g^4 with size $2g^2$ for every $g \geq 4$ and $g \not\equiv 2 \pmod{4}$.*

3.3 Some $[2, 2]$ -GDC(6)s

Lemma 3.7 *There exists a $[2, 2]$ -GDC(6) of type 2^{10} with size 60.*

Proof: Let $X = \mathbb{Z}_{20}$, and $\mathcal{G} = \{\{i, i + 10\} : 0 \leq i \leq 9\}$. Then $(X, \mathcal{G}, \mathcal{C})$ is a $[2, 2]$ -GDC(6) of type 2^{10} , where \mathcal{C} is obtained by developing the elements of \mathbb{Z}_{20} in the codewords $\langle 0, 5, 3, 7 \rangle, \langle 0, 4, 1, 13 \rangle, \langle 0, 8, 14, 19 \rangle + 1 \pmod{20}$. \square

Lemma 3.8 *There exists a $[2, 2]$ -GDC(6) of type 6^t with size $6t(t - 1)$ for each $5 \leq t \leq 11$.*

Proof: For $t \in \{5, 8\}$, let $X_t = \mathbb{Z}_{6t}$, and $\mathcal{G}_t = \{\{i, i + t, i + 2t, i + 3t, i + 4t, i + 5t\} : 0 \leq i \leq t - 1\}$. Then $(X_t, \mathcal{G}_t, \mathcal{C}_t)$ is a $[2, 2]$ -GDC(6) of type 6^t and size $6t(t - 1)$, where \mathcal{C}_t is obtained by developing the elements of \mathbb{Z}_{6t} in the following codewords $+1 \pmod{6t}$.

$t = 5$: $\langle 0, 24, 1, 13 \rangle \langle 0, 9, 8, 11 \rangle \langle 0, 3, 17, 26 \rangle \langle 0, 12, 4, 28 \rangle$

$t = 8$: $\langle 0, 18, 13, 3 \rangle \langle 0, 28, 21, 25 \rangle \langle 0, 10, 22, 36 \rangle \langle 0, 46, 27, 9 \rangle \langle 0, 14, 31, 37 \rangle \langle 0, 6, 5, 7 \rangle \langle 0, 4, 19, 39 \rangle$

For $t = 6$, let $X_6 = \mathbb{Z}_{12} \times I_3$, and $\mathcal{G}_6 = \{\{(i, 0), (i + 6, 0), (i, 1), (i + 6, 1), (i, 2), (i + 6, 2)\} : 0 \leq i \leq 5\}$. Then $(X_6, \mathcal{G}_6, \mathcal{C}_6)$ is a $[2, 2]$ -GDC(6) of type 6^6 and size 180, where \mathcal{C}_6 is obtained by developing the elements of $\mathbb{Z}_{12} \times I_3$ in the following codewords $(+1 \pmod{12}), (-)$.

$$\begin{array}{cccccc} \langle 0_1, 9_1, 10_0, 5_0 \rangle & \langle 0_0, 5_0, 8_0, 9_0 \rangle & \langle 0_0, 2_0, 7_2, 4_2 \rangle & \langle 0_0, 11_0, 7_1, 10_1 \rangle & \langle 0_2, 2_2, 4_1, 7_0 \rangle \\ \langle 0_2, 9_2, 1_0, 11_0 \rangle & \langle 0_1, 3_0, 4_2, 2_2 \rangle & \langle 0_2, 5_2, 8_0, 1_1 \rangle & \langle 0_1, 11_0, 8_2, 7_2 \rangle & \langle 0_1, 2_1, 4_0, 7_1 \rangle \\ \langle 0_1, 7_0, 10_2, 5_2 \rangle & \langle 0_2, 9_1, 5_1, 8_2 \rangle & \langle 0_1, 9_2, 1_2, 4_1 \rangle & \langle 0_1, 9_0, 11_1, 1_1 \rangle & \langle 0_2, 11_2, 10_1, 9_0 \rangle \end{array}$$

For $t \in \{7, 9, 11\}$, the desired GDCs are obtained from Lemma 3.3. For $t = 10$, inflate a $[2, 2]$ -GDC(6) of type 2^{10} with weight 3 to obtain the desired code. \square

Lemma 3.9 *There exists a $[2, 2]$ -GDC(6) of type g^u with size $u(u-1)g^2/6$ for each $(g, u) \in \{(3, 7), (3, 11), (3, 13), (10, 7), (18, 4)\}$.*

Proof: For each $[2, 2]$ -GDC(6) of type g^u , let $X = \mathbb{Z}_{gu}$, and $\mathcal{G} = \{\{i, i+u, i+2u, \dots, i+(g-1)u\} : 0 \leq i \leq u-1\}$. Then $(X, \mathcal{G}, \mathcal{C})$ is a $[2, 2]$ -GDC(6) of type g^u , where \mathcal{C} is obtained by developing the elements of \mathbb{Z}_{gu} in the following codewords under the automorphism group as below.

3^7 : $+1 \pmod{21}$

$$\langle 7, 3, 18, 13 \rangle \quad \langle 4, 3, 12, 16 \rangle \quad \langle 0, 5, 2, 3 \rangle$$

3^{11} : $+1 \pmod{33}$

$$\langle 25, 15, 0, 7 \rangle \quad \langle 13, 27, 15, 25 \rangle \quad \langle 0, 7, 1, 6 \rangle \quad \langle 10, 5, 1, 14 \rangle \quad \langle 2, 15, 18, 32 \rangle$$

3^{13} : $+1 \pmod{39}$

$$\langle 20, 2, 25, 10 \rangle \quad \langle 21, 6, 23, 17 \rangle \quad \langle 6, 12, 5, 26 \rangle \quad \langle 30, 3, 28, 19 \rangle \quad \langle 21, 24, 31, 4 \rangle \quad \langle 0, 9, 1, 4 \rangle$$

10^7 : $+1 \pmod{70}$

$$\begin{aligned} &\langle 0, 31, 50, 5 \rangle \quad \langle 0, 61, 24, 46 \rangle \quad \langle 0, 25, 6, 68 \rangle \quad \langle 0, 32, 16, 52 \rangle \quad \langle 0, 3, 2, 40 \rangle \quad \langle 0, 34, 47, 64 \rangle \\ &\langle 0, 48, 1, 60 \rangle \quad \langle 0, 17, 27, 58 \rangle \quad \langle 0, 11, 26, 29 \rangle \quad \langle 0, 62, 57, 66 \rangle \end{aligned}$$

18^4 : $+2 \pmod{72}$

$$\begin{aligned} &\langle 1, 3, 8, 22 \rangle \quad \langle 0, 34, 1, 15 \rangle \quad \langle 1, 7, 16, 54 \rangle \quad \langle 0, 2, 21, 71 \rangle \quad \langle 0, 9, 54, 63 \rangle \quad \langle 0, 10, 5, 47 \rangle \\ &\langle 0, 6, 51, 41 \rangle \quad \langle 0, 7, 18, 25 \rangle \quad \langle 43, 5, 2, 28 \rangle \quad \langle 43, 1, 4, 14 \rangle \quad \langle 0, 26, 11, 13 \rangle \quad \langle 1, 15, 50, 56 \rangle \\ &\langle 0, 30, 23, 61 \rangle \quad \langle 0, 14, 43, 17 \rangle \quad \langle 0, 22, 49, 55 \rangle \quad \langle 1, 23, 60, 18 \rangle \quad \langle 1, 27, 26, 28 \rangle \quad \langle 1, 11, 62, 40 \rangle \end{aligned}$$

□

Lemma 3.10 *There exists a $[2, 2]$ -GDC(6) of type $6^t 3^1$ with size $6t^2$ for each $t = 4$ or $6 \leq t \leq 11$.*

Proof: Let $X_t = I_{6t+3}$, and $\mathcal{G}_t = \{\{i, i+t, i+2t, \dots, i+5t\} : 0 \leq i \leq t-1\} \cup \{\{6t, 6t+1, 6t+2\}\}$. Then $(X_t, \mathcal{G}_t, \mathcal{C}_t)$ is a $[2, 2]$ -GDC(6) of type $6^t 3^1$, where \mathcal{C}_4 is obtained by developing the following codewords under the automorphism group $G = \langle (0 \ 2 \ 4 \ \dots \ 22)(1 \ 3 \ 5 \ \dots \ 23)(24 \ 25 \ 26) \rangle$, and \mathcal{C}_t for $6 \leq t \leq 11$ is obtained by developing the following codewords under the automorphism group $G = \langle (0 \ 1 \ 2 \ \dots \ 6t-1)(6t \ 6t+1 \ 6t+2) \rangle$.

$t = 4$:

$$\begin{aligned} &\langle 1, 25, 8, 6 \rangle \quad \langle 0, 10, 24, 23 \rangle \quad \langle 1, 7, 10, 20 \rangle \quad \langle 0, 26, 19, 17 \rangle \\ &\langle 0, 22, 7, 1 \rangle \quad \langle 1, 11, 12, 26 \rangle \quad \langle 0, 6, 21, 11 \rangle \quad \langle 1, 3, 18, 0 \rangle \end{aligned}$$

$t = 6$:

$$\langle 0, 34, 27, 5 \rangle \quad \langle 0, 28, 3, 13 \rangle \quad \langle 0, 10, 19, 35 \rangle \quad \langle 0, 20, 17, 15 \rangle \quad \langle 0, 37, 32, 4 \rangle \quad \langle 0, 14, 1, 38 \rangle$$

$t = 7$:

$$\begin{aligned} &\langle 0, 26, 25, 44 \rangle \quad \langle 0, 8, 12, 38 \rangle \quad \langle 0, 43, 20, 22 \rangle \quad \langle 0, 40, 1, 9 \rangle \quad \langle 0, 6, 23, 33 \rangle \quad \langle 0, 18, 31, 37 \rangle \\ &\langle 0, 10, 15, 39 \rangle \end{aligned}$$

$t = 8$:

$$\begin{aligned} &\langle 0, 10, 12, 46 \rangle \quad \langle 0, 28, 5, 23 \rangle \quad \langle 0, 26, 41, 45 \rangle \quad \langle 0, 18, 3, 9 \rangle \quad \langle 0, 14, 13, 49 \rangle \quad \langle 0, 42, 1, 29 \rangle \\ &\langle 0, 48, 11, 37 \rangle \quad \langle 0, 4, 21, 31 \rangle \end{aligned}$$

$t = 9$:

$$\begin{aligned} &\langle 0, 52, 5, 21 \rangle \quad \langle 0, 54, 49, 53 \rangle \quad \langle 0, 38, 3, 51 \rangle \quad \langle 0, 14, 1, 47 \rangle \quad \langle 0, 8, 43, 55 \rangle \quad \langle 0, 34, 12, 22 \rangle \\ &\langle 0, 6, 17, 37 \rangle \quad \langle 0, 44, 15, 29 \rangle \quad \langle 0, 4, 28, 30 \rangle \end{aligned}$$

$t = 10$:

$$\begin{aligned} &\langle 0, 12, 36, 37 \rangle \quad \langle 0, 54, 45, 57 \rangle \quad \langle 0, 8, 29, 35 \rangle \quad \langle 0, 34, 23, 61 \rangle \quad \langle 0, 62, 19, 41 \rangle \quad \langle 0, 4, 9, 17 \rangle \\ &\langle 0, 1, 15, 47 \rangle \quad \langle 0, 58, 16, 42 \rangle \quad \langle 0, 28, 7, 11 \rangle \quad \langle 0, 22, 53, 55 \rangle \end{aligned}$$

$t = 11$:

$$\begin{aligned} &\langle 0, 10, 1, 13 \rangle \quad \langle 0, 14, 43, 68 \rangle \quad \langle 0, 12, 30, 38 \rangle \quad \langle 0, 6, 15, 47 \rangle \quad \langle 0, 8, 59, 61 \rangle \quad \langle 0, 21, 49, 63 \rangle \\ &\langle 0, 67, 7, 17 \rangle \quad \langle 0, 16, 36, 40 \rangle \quad \langle 0, 64, 25, 46 \rangle \quad \langle 0, 34, 5, 65 \rangle \quad \langle 0, 4, 23, 39 \rangle \end{aligned}$$

□

Lemma 3.11 *There exists a $[2, 2]$ -GDC(6) of type $6^t 9^1$ with size $6t(t + 2)$ for each $t \in \{6, 7, 8\}$.*

Proof: Let $X_t = I_{6t+9}$, and $\mathcal{G}_t = \{\{i, i + t, i + 2t, \dots, i + 5t\} : 0 \leq i \leq t - 1\} \cup \{\{6t, 6t + 1, 6t + 2, \dots, 6t + 8\}\}$. Then $(X_t, \mathcal{G}_t, \mathcal{C}_t)$ is a $[2, 2]$ -GDC(6) of type $6^t 9^1$, where \mathcal{C}_t is obtained by developing the following codewords under the automorphism group $G = \langle (0 \ 1 \ 2 \ \dots \ 6t - 1)(6t \ 6t + 1 \ 6t + 2 \ 6t + 3 \ 6t + 4 \ 6t + 5)(6t + 6 \ 6t + 7 \ 6t + 8) \rangle$.

$t = 6$:

$$\begin{aligned} &\langle 0, 36, 5, 15 \rangle \quad \langle 0, 22, 13, 29 \rangle \quad \langle 0, 34, 21, 38 \rangle \quad \langle 24, 4, 7, 37 \rangle \\ &\langle 0, 10, 9, 11 \rangle \quad \langle 0, 43, 17, 31 \rangle \quad \langle 0, 41, 33, 25 \rangle \quad \langle 0, 8, 4, 44 \rangle \end{aligned}$$

$t = 7$:

$$\begin{aligned} &\langle 0, 36, 34, 10 \rangle \quad \langle 0, 45, 8, 39 \rangle \quad \langle 0, 23, 25, 49 \rangle \quad \langle 0, 44, 3, 22 \rangle \\ &\langle 0, 24, 41, 12 \rangle \quad \langle 0, 5, 46, 32 \rangle \quad \langle 0, 29, 33, 42 \rangle \quad \langle 0, 48, 38, 1 \rangle \\ &\langle 0, 11, 20, 26 \rangle \end{aligned}$$

$t = 8$:

$$\begin{array}{cccc} \langle 0, 10, 3, 15 \rangle & \langle 0, 14, 54, 37 \rangle & \langle 0, 49, 27, 29 \rangle & \langle 0, 36, 31, 1 \rangle \\ \langle 0, 18, 9, 35 \rangle & \langle 0, 56, 47, 19 \rangle & \langle 0, 22, 51, 33 \rangle & \langle 0, 46, 42, 4 \rangle \\ \langle 0, 48, 7, 21 \rangle & \langle 0, 20, 52, 45 \rangle & & \end{array}$$

□

Lemma 3.12 *There exists a $[2, 2]$ -GDC(6) of type $g^u m^1$ with size $(g^2 u(u-1) + 2gum)/6$ for each $(g, u, m) \in \{(9, 5, 9), (9, 5, 15), (18, 4, 6), (18, 4, 12), (18, 6, 33), (24, 4, 6), (24, 4, 9)\}$.*

Proof: For each $[2, 2]$ -GDC(6) of type $g^u m^1$, let $X = I_{gu+m}$, and $\mathcal{G} = \{\{0, u, 2u, \dots, (g-1)u\} + i : 0 \leq i \leq u-1\} \cup \{\{gu, gu+1, gu+2, \dots, gu+m-1\}\}$. Then $(X, \mathcal{G}, \mathcal{C})$ is the desired $[2, 2]$ -GDC(6), where \mathcal{C} is obtained by developing the following codewords under the automorphism group as below.

$$9^5 9^1: G = \langle (0 \ 1 \ 2 \ \dots \ 44)(45 \ 46 \ \dots \ 53) \rangle$$

$$\begin{array}{cccccc} \langle 2, 45, 43, 1 \rangle & \langle 0, 9, 17, 28 \rangle & \langle 24, 21, 42, 3 \rangle & \langle 4, 45, 30, 8 \rangle & \langle 8, 42, 45, 40 \rangle & \langle 17, 40, 24, 33 \rangle \\ \langle 1, 7, 45, 38 \rangle & \langle 0, 12, 45, 14 \rangle & \langle 14, 45, 27, 15 \rangle & & & \end{array}$$

$$9^5 15^1: G = \langle (0 \ 1 \ 2 \ \dots \ 44)(45 \ 46 \ \dots \ 59) \rangle$$

$$\begin{array}{cccccc} \langle 45, 35, 18, 27 \rangle & \langle 45, 13, 9, 16 \rangle & \langle 45, 41, 4, 17 \rangle & \langle 24, 15, 41, 45 \rangle & \langle 45, 22, 40, 6 \rangle & \langle 0, 11, 42, 44 \rangle \\ \langle 18, 16, 22, 45 \rangle & \langle 8, 21, 35, 45 \rangle & \langle 4, 27, 43, 45 \rangle & \langle 10, 17, 29, 45 \rangle & \langle 45, 14, 15, 38 \rangle & \end{array}$$

$$18^4 6^1: G = \langle (0 \ 1 \ 2 \ \dots \ 71)(72 \ 73 \ 74 \ 75 \ 76 \ 77) \rangle$$

$$\begin{array}{cccccc} \langle 0, 26, 9, 19 \rangle & \langle 0, 54, 21, 59 \rangle & \langle 0, 50, 53, 76 \rangle & \langle 0, 72, 29, 51 \rangle & \langle 0, 2, 13, 71 \rangle & \langle 0, 30, 1, 7 \rangle \\ \langle 0, 34, 61, 73 \rangle & \langle 0, 66, 17, 35 \rangle & \langle 0, 10, 25, 67 \rangle & \langle 0, 58, 31, 33 \rangle & \langle 0, 77, 37, 63 \rangle & \end{array}$$

$$18^4 12^1: G = \langle (0 \ 1 \ 2 \ \dots \ 71)(72 \ 73 \ 74 \ 75 \ 76 \ 77) (78 \ 79 \ 80 \ 81 \ 82 \ 83) \rangle$$

$$\begin{array}{cccccc} \langle 0, 46, 7, 72 \rangle & \langle 0, 77, 57, 71 \rangle & \langle 0, 34, 63, 79 \rangle & \langle 0, 22, 49, 78 \rangle & \langle 0, 14, 1, 31 \rangle & \langle 0, 10, 47, 65 \rangle \\ \langle 0, 83, 9, 11 \rangle & \langle 0, 18, 23, 61 \rangle & \langle 0, 42, 39, 45 \rangle & \langle 0, 82, 21, 67 \rangle & \langle 0, 6, 19, 41 \rangle & \langle 0, 70, 51, 73 \rangle \\ \langle 0, 76, 15, 25 \rangle & & & & & \end{array}$$

$$18^6 33^1: G = \langle (0 \ 1 \ 2 \ \dots \ 107)(108 \ 109 \ \dots \ 113) (114 \ 115 \ \dots \ 119)(120 \ 121 \ \dots \ 125) (126 \ 127 \ \dots \ 131) (132 \ 133 \ \dots \ 140) \rangle$$

$$\begin{array}{cccccc} \langle 0, 138, 53, 97 \rangle & \langle 0, 20, 17, 131 \rangle & \langle 0, 82, 7, 128 \rangle & \langle 0, 70, 92, 139 \rangle & \langle 0, 52, 27, 108 \rangle & \langle 0, 120, 4, 9 \rangle \\ \langle 0, 121, 68, 81 \rangle & \langle 0, 95, 58, 140 \rangle & \langle 0, 47, 43, 63 \rangle & \langle 0, 74, 67, 118 \rangle & \langle 0, 73, 51, 124 \rangle & \langle 0, 8, 40, 133 \rangle \\ \langle 0, 115, 37, 99 \rangle & \langle 0, 136, 28, 29 \rangle & \langle 0, 39, 25, 98 \rangle & \langle 0, 117, 10, 31 \rangle & \langle 0, 127, 11, 80 \rangle & \langle 0, 111, 19, 93 \rangle \\ \langle 0, 87, 89, 125 \rangle & \langle 0, 137, 23, 49 \rangle & \langle 0, 44, 15, 85 \rangle & \langle 0, 62, 45, 109 \rangle & \langle 0, 126, 14, 75 \rangle & \langle 0, 112, 57, 65 \rangle \\ \langle 0, 107, 76, 119 \rangle & \langle 0, 103, 50, 106 \rangle & & & & \end{array}$$

$$24^4 6^1: G = \langle (0 \ 1 \ 2 \ \dots \ 95)(96 \ 97 \ 98 \ 99 \ 100 \ 101) \rangle$$

$$\begin{array}{llllll} \langle 0, 86, 25, 75 \rangle & \langle 0, 82, 81, 97 \rangle & \langle 0, 42, 23, 29 \rangle & \langle 0, 101, 63, 65 \rangle & \langle 0, 30, 21, 47 \rangle & \langle 0, 62, 5, 19 \rangle \\ \langle 0, 38, 11, 89 \rangle & \langle 0, 26, 3, 98 \rangle & \langle 0, 18, 27, 49 \rangle & \langle 0, 100, 57, 67 \rangle & \langle 0, 74, 37, 71 \rangle & \langle 0, 2, 15, 45 \rangle \\ \langle 0, 46, 41, 79 \rangle & \langle 0, 6, 7, 61 \rangle & & & & \end{array}$$

$$24^4 9^1: G = \langle (0 \ 1 \ 2 \ \dots \ 95)(96 \ 97 \ 98)(99 \ 100 \ 101) (102 \ 103 \ 104) \rangle$$

$$\begin{array}{llllll} \langle 0, 82, 15, 73 \rangle & \langle 0, 38, 3, 49 \rangle & \langle 0, 34, 23, 101 \rangle & \langle 0, 10, 81, 7 \rangle & \langle 0, 26, 19, 25 \rangle & \langle 0, 54, 41, 59 \rangle \\ \langle 0, 18, 45, 31 \rangle & \langle 0, 6, 39, 69 \rangle & \langle 0, 22, 17, 104 \rangle & \langle 0, 30, 51, 9 \rangle & \langle 0, 96, 43, 77 \rangle & \langle 0, 46, 47, 98 \rangle \\ \langle 0, 99, 53, 79 \rangle & \langle 0, 2, 57, 67 \rangle & \langle 0, 102, 35, 37 \rangle & & & \end{array}$$

□

Let $P = [9, 19] \cup [21, 23] \cup [26, 28] \cup [31, 33]$.

Lemma 3.13 *For each $t \geq 9$ and $t \notin P$, there exists a $[2, 2]$ -GDC(6) of type $24^i 30^j 36^k 42^l 48^m$, where i, j, k, l, m are integers such that $4i + 5j + 6k + 7l + 8m = t$.*

Proof: For each $t \geq 9$ and $t \notin P$, take a $(t + 1, \{5, 6, 7, 8, 9\}, 1)$ -PBD from Theorem 2.6, and remove one point from the point set to obtain a $\{5, 6, 7, 8, 9\}$ -GDD of type $4^i 5^j 6^k 7^l 8^m$ with $4i + 5j + 6k + 7l + 8m = t$. Apply the Fundamental Construction with weight 6 to this GDD and input $[2, 2]$ -GDC(6)s of type 6^u for $u \in \{5, 6, 7, 8, 9\}$ from Lemma 3.8 to obtain a $[2, 2]$ -GDC(6) of type $24^i 30^j 36^k 42^l 48^m$. □

Lemma 3.14 *The following $[2, 2]$ -GDC(6)s all exist:*

- i) type 18^u and size $54u(u - 1)$ for $u \in \{4, 5, 6, 7, 9, 11\}$;
- ii) type 24^u and size $96u(u - 1)$ for $u \in \{4, 7, 8\}$;
- iii) type $24^u 36^1$ and size $96u(u + 2)$ for $u \in \{4, 5\}$;
- iv) type $18^8 42^1$ and size 5040;
- v) type $30^4 18^1$ and size 2520;
- vi) type $24^4 18^1$ and size 1728.

Proof: For i), a $[2, 2]$ -GDC(6) of type 18^4 is constructed directly in Lemma 3.9. For each $t \in \{5, 6, 7, 9, 11\}$, take a $[2, 2]$ -GDC(6) of type 6^t from Lemma 3.8, and inflate it with weight 3 to obtain a $[2, 2]$ -GDC(6) of type 18^t .

For ii), the required GDCs are obtained from Lemma 3.3.

For iii), the required GDCs are obtained by applying the Fundamental Construction with weight 4 to a 4-GDD of type $6^u 9^1$ (see [27, Theorem 1.6]). The input $[2, 2]$ -GDC(6)s of type 4^4 come from Lemma 3.6.

For iv), take a 5-GDD of type $3^8 7^1$ obtained by completing the parallel classes of a 4-RGDD of type 3^8 (see [33]). Apply the Fundamental Construction with weight 6 to obtain a $[2, 2]$ -GDC(6) of type $18^8 42^1$. The input $[2, 2]$ -GDC(6)s of type 6^5 come from Lemma 3.8.

For v), take a TD(5, 5) and apply the Fundamental Construction, giving weight 6 to each point in the first four groups and one point in the last group and weight 3 to each of the remaining points. Noting that there exist $[2, 2]$ -GDC(6)s of types 6^5 and $6^4 3^1$ by Lemmas 3.8 and 3.10, we get a $[2, 2]$ -GDC(6) of type $30^4 18^1$.

For vi), take a TD(5, 4) and apply the Fundamental Construction, giving weight 6 to each point in the first four groups and two points in the last group and weight 3 to each of the remaining points. Noting that there exist $[2, 2]$ -GDC(6)s of types 6^5 and $6^4 3^1$ by Lemmas 3.8 and 3.10, we obtain a $[2, 2]$ -GDC(6) of type $24^4 18^1$. \square

3.4 Cases of Length $n \equiv 0, 1 \pmod{6}$

Lemma 3.15 $A_3(7, 6, [2, 2]) = 3$, $A_3(13, 6, [2, 2]) \geq 21$.

Proof: For $n = 7$, see Table 1.

For $n = 13$, the required code is constructed on I_{13} with codewords as below.

$$\begin{array}{cccccc} \langle 9, 3, 5, 6 \rangle & \langle 11, 7, 1, 6 \rangle & \langle 9, 12, 11, 7 \rangle & \langle 2, 5, 0, 7 \rangle & \langle 1, 4, 7, 10 \rangle & \langle 9, 10, 0, 1 \rangle \\ \langle 0, 7, 8, 9 \rangle & \langle 3, 12, 1, 4 \rangle & \langle 0, 12, 10, 5 \rangle & \langle 0, 1, 2, 3 \rangle & \langle 10, 5, 8, 4 \rangle & \langle 4, 11, 2, 9 \rangle \\ \langle 1, 6, 9, 12 \rangle & \langle 0, 6, 4, 11 \rangle & \langle 10, 7, 12, 2 \rangle & \langle 8, 6, 3, 7 \rangle & \langle 1, 8, 11, 5 \rangle & \langle 2, 12, 6, 8 \rangle \\ \langle 8, 4, 0, 12 \rangle & \langle 2, 3, 10, 11 \rangle & \langle 5, 11, 3, 12 \rangle & & & \end{array}$$

\square

Lemma 3.16 $A_3(6t + 1, 6, [2, 2]) = U(6t + 1, 6, [2, 2])$ for each $t \in [3, 11] \cup \{13, 14, 17\}$.

Proof: For each $t \in [3, 11] \cup \{13, 14, 17\}$ and $t \neq 4$, let $X_t = \mathbb{Z}_{6t+1}$. Then (X_t, \mathcal{C}_t) is the desired optimal $(6t + 1, 6, [2, 2])_3$ -code, where \mathcal{C}_t is obtained by developing the elements of \mathbb{Z}_{6t+1} in the codewords listed in Table 2 $+1 \pmod{6t+1}$.

Let $X_4 = \mathbb{Z}_5 \times \mathbb{Z}_5$. Then (X_4, \mathcal{C}_4) is the desired optimal $(25, 6, [2, 2])_3$ -code, where \mathcal{C}_4 is obtained by developing the elements of $\mathbb{Z}_5 \times \mathbb{Z}_5$ in the codewords listed in Table 2 $(+1 \pmod{5}, +1 \pmod{5})$. \square

Lemma 3.17 $A_3(6t + 1, 6, [2, 2]) = U(6t + 1, 6, [2, 2])$ for each $t \geq 12$ and $t \notin \{13, 14, 17\}$.

Table 2: Base Codewords of Small Optimal $(6t + 1, 6, [2, 2])_3$ -Codes in Lemma 3.16

t	Codewords
3	$\langle 0, 1, 4, 16 \rangle \langle 0, 7, 9, 17 \rangle \langle 0, 8, 13, 14 \rangle$
4	$\langle 0_3, 2_0, 3_0, 4_3 \rangle \langle 0_2, 2_3, 2_1, 0_4 \rangle \langle 0_2, 1_0, 1_4, 0_3 \rangle \langle 0_0, 1_1, 2_0, 4_1 \rangle$
5	$\langle 0, 3, 11, 23 \rangle \langle 0, 7, 2, 5 \rangle \langle 0, 6, 15, 22 \rangle \langle 0, 14, 4, 10 \rangle \langle 0, 12, 13, 30 \rangle$
6	$\langle 0, 23, 27, 35 \rangle \langle 0, 8, 11, 17 \rangle \langle 0, 5, 25, 1 \rangle \langle 0, 6, 22, 36 \rangle \langle 0, 18, 2, 7 \rangle \langle 0, 13, 10, 28 \rangle$
7	$\langle 0, 35, 26, 23 \rangle \langle 0, 16, 33, 37 \rangle \langle 0, 29, 22, 38 \rangle \langle 0, 3, 10, 18 \rangle \langle 0, 30, 11, 12 \rangle \langle 0, 1, 6, 20 \rangle \langle 0, 4, 2, 32 \rangle$
8	$\langle 0, 36, 40, 45 \rangle \langle 0, 28, 14, 47 \rangle \langle 0, 42, 10, 32 \rangle \langle 0, 33, 25, 18 \rangle \langle 0, 44, 15, 26 \rangle \langle 0, 11, 48, 12 \rangle \langle 0, 43, 23, 2 \rangle \langle 0, 22, 3, 46 \rangle$
9	$\langle 0, 8, 1, 21 \rangle \langle 0, 43, 19, 42 \rangle \langle 0, 32, 34, 39 \rangle \langle 0, 20, 30, 45 \rangle \langle 0, 28, 9, 26 \rangle \langle 0, 17, 41, 14 \rangle \langle 0, 15, 6, 18 \rangle \langle 0, 5, 16, 49 \rangle$
10	$\langle 0, 27, 36, 41 \rangle \langle 0, 11, 21, 39 \rangle \langle 0, 3, 22, 26 \rangle \langle 0, 1, 47, 53 \rangle \langle 0, 5, 35, 37 \rangle \langle 0, 17, 24, 25 \rangle \langle 0, 2, 15, 42 \rangle \langle 0, 4, 33, 16 \rangle$
11	$\langle 0, 20, 33, 44 \rangle \langle 0, 1, 32, 41 \rangle \langle 0, 10, 49, 53 \rangle \langle 0, 30, 48, 51 \rangle \langle 0, 3, 28, 65 \rangle \langle 0, 7, 26, 36 \rangle \langle 0, 11, 16, 17 \rangle \langle 0, 8, 23, 35 \rangle$
13	$\langle 0, 11, 54, 56 \rangle \langle 0, 5, 49, 60 \rangle \langle 0, 18, 35, 64 \rangle \langle 0, 63, 66, 76 \rangle \langle 0, 29, 65, 71 \rangle \langle 0, 10, 24, 33 \rangle \langle 0, 20, 51, 67 \rangle \langle 0, 7, 15, 19 \rangle$
14	$\langle 0, 9, 41, 53 \rangle \langle 0, 1, 50, 83 \rangle \langle 0, 45, 47, 63 \rangle \langle 0, 46, 61, 70 \rangle \langle 0, 4, 25, 59 \rangle \langle 0, 17, 27, 28 \rangle \langle 0, 52, 57, 65 \rangle \langle 0, 8, 56, 75 \rangle$
17	$\langle 0, 16, 74, 80 \rangle \langle 0, 19, 22, 62 \rangle \langle 0, 6, 29, 36 \rangle \langle 0, 7, 38, 42 \rangle \langle 0, 12, 26, 72 \rangle \langle 0, 34, 54, 71 \rangle$
	$\langle 0, 20, 62, 63 \rangle \langle 0, 10, 58, 88 \rangle \langle 0, 3, 59, 95 \rangle \langle 0, 17, 24, 50 \rangle \langle 0, 1, 76, 97 \rangle \langle 0, 12, 46, 66 \rangle \langle 0, 26, 39, 41 \rangle \langle 0, 14, 69, 79 \rangle$
	$\langle 0, 36, 64, 81 \rangle \langle 0, 74, 85, 99 \rangle \langle 0, 5, 40, 49 \rangle \langle 0, 19, 87, 90 \rangle \langle 0, 9, 47, 70 \rangle \langle 0, 21, 53, 72 \rangle \langle 0, 30, 52, 57 \rangle \langle 0, 23, 31, 60 \rangle$
	$\langle 0, 2, 6, 18 \rangle$

Proof: For each $t \geq 12$ and $t \notin P$, take a $[2, 2]$ -GDC(6) of type $24^i 30^j 36^k 42^l 48^m$ with $4i + 5j + 6k + 7l + 8m = t$ from Lemma 3.13. Adjoin one ideal point, and fill in the groups together with the ideal point with optimal codes of small lengths from Lemma 3.16 to obtain the desired code.

For each $t \in \{12, 15, 16, 18, 19, 21, 22, 23, 26, 27, 28, 31, 32, 33\}$, take a $[2, 2]$ -GDC(6) constructed in Lemma 3.14. Adjoin one ideal point, and fill in the groups together with the ideal point with optimal codes of small lengths from Lemma 3.16 to obtain the desired code. \square

Combining the above lemmas, we obtain the following result for $n \equiv 1 \pmod{6}$.

Theorem 3.18 $A_3(7, 6, [2, 2]) = 3$, $A_3(13, 6, [2, 2]) \geq 21$, $A_3(6t + 1, 6, [2, 2]) = U(6t + 1, 6, [2, 2])$ for each $t \geq 3$;

For $n \equiv 0 \pmod{6}$, we have the following result.

Theorem 3.19 $A_3(6t, 6, [2, 2]) = U(6t, 6, [2, 2])$ for each $t \geq 1$.

Proof: For $t = 1$, see Table 1. For $t = 2$, let $X = \mathbb{Z}_{12}$. Then the code \mathcal{C} is obtained by developing the elements of \mathbb{Z}_{12} in the following codewords $+2 \pmod{12}$.

$$\langle 0, 2, 8, 5 \rangle \quad \langle 0, 9, 7, 11 \rangle \quad \langle 1, 5, 0, 10 \rangle$$

For each $t \geq 3$, remove one point and related codewords from an optimal $(6t + 1, 6, [2, 2])_3$ -code from Theorem 3.18 to get the desired code. \square

Table 3: Base Codewords of Small $[2, 2]$ -GDC(6)s of Type 2^{3t+1} in Lemma 3.21

t	Codewords							
4	$\langle 0, 4, 3, 11 \rangle \langle 0, 5, 6, 15 \rangle \langle 0, 9, 2, 23 \rangle \langle 0, 8, 20, 24 \rangle$							
5	$\langle 0, 28, 9, 29 \rangle \langle 1, 7, 2, 12 \rangle \langle 0, 15, 24, 7 \rangle \langle 0, 10, 30, 12 \rangle \langle 0, 18, 23, 21 \rangle \langle 1, 14, 22, 9 \rangle \langle 1, 21, 4, 8 \rangle \langle 0, 26, 25, 11 \rangle$ $\langle 1, 15, 11, 5 \rangle \langle 1, 3, 0, 26 \rangle$							
6	$\langle 0, 7, 20, 5 \rangle \langle 0, 21, 11, 27 \rangle \langle 0, 23, 14, 35 \rangle \langle 0, 30, 24, 25 \rangle \langle 0, 22, 18, 26 \rangle \langle 0, 1, 3, 10 \rangle$							
7	$\langle 0, 5, 21, 33 \rangle \langle 0, 32, 34, 42 \rangle \langle 0, 14, 23, 29 \rangle \langle 0, 6, 7, 25 \rangle \langle 0, 36, 40, 35 \rangle \langle 0, 20, 17, 3 \rangle \langle 0, 18, 11, 31 \rangle$							
8	$\langle 0, 40, 29, 34 \rangle \langle 0, 30, 2, 11 \rangle \langle 0, 37, 3, 33 \rangle \langle 0, 45, 12, 27 \rangle \langle 0, 35, 21, 8 \rangle \langle 0, 49, 18, 42 \rangle \langle 0, 9, 7, 6 \rangle \langle 0, 24, 28, 38 \rangle$							
9	$\langle 0, 39, 33, 45 \rangle \langle 0, 12, 48, 53 \rangle \langle 0, 22, 35, 9 \rangle \langle 0, 5, 3, 37 \rangle \langle 0, 29, 31, 47 \rangle \langle 0, 4, 25, 42 \rangle \langle 0, 1, 11, 15 \rangle \langle 0, 16, 23, 24 \rangle$ $\langle 0, 26, 19, 46 \rangle$							
10	$\langle 0, 9, 38, 57 \rangle \langle 0, 22, 30, 33 \rangle \langle 0, 15, 39, 5 \rangle \langle 0, 20, 12, 21 \rangle \langle 0, 58, 23, 45 \rangle \langle 0, 3, 2, 17 \rangle \langle 0, 25, 35, 41 \rangle \langle 0, 19, 51, 55 \rangle$ $\langle 0, 6, 13, 50 \rangle \langle 0, 28, 46, 26 \rangle$							
11	$\langle 0, 55, 33, 19 \rangle \langle 0, 37, 65, 49 \rangle \langle 0, 52, 8, 63 \rangle \langle 0, 67, 60, 2 \rangle \langle 0, 10, 4, 35 \rangle \langle 0, 59, 66, 48 \rangle \langle 0, 17, 44, 53 \rangle \langle 0, 50, 22, 21 \rangle$ $\langle 0, 23, 38, 64 \rangle \langle 0, 26, 5, 56 \rangle \langle 0, 14, 20, 43 \rangle$							
14	$\langle 0, 1, 20, 31 \rangle \langle 0, 7, 47, 68 \rangle \langle 0, 11, 15, 78 \rangle \langle 0, 14, 71, 80 \rangle \langle 0, 23, 82, 83 \rangle \langle 0, 5, 29, 41 \rangle \langle 0, 10, 38, 55 \rangle \langle 0, 2, 46, 56 \rangle$ $\langle 0, 9, 35, 42 \rangle \langle 0, 12, 18, 70 \rangle \langle 0, 17, 25, 39 \rangle \langle 0, 34, 37, 50 \rangle \langle 0, 13, 62, 64 \rangle \langle 0, 21, 48, 53 \rangle$							
17	$\langle 0, 1, 81, 86 \rangle \langle 0, 4, 37, 58 \rangle \langle 0, 8, 65, 74 \rangle \langle 0, 12, 59, 76 \rangle \langle 0, 22, 71, 72 \rangle \langle 0, 51, 78, 90 \rangle \langle 0, 3, 19, 29 \rangle \langle 0, 7, 75, 77 \rangle$ $\langle 0, 2, 40, 62 \rangle \langle 0, 5, 11, 46 \rangle \langle 0, 9, 32, 88 \rangle \langle 0, 17, 30, 61 \rangle \langle 0, 31, 45, 98 \rangle \langle 0, 69, 89, 93 \rangle \langle 0, 10, 25, 28 \rangle \langle 0, 21, 55, 63 \rangle$ $\langle 0, 48, 84, 91 \rangle$							

3.5 Case of Length $n \equiv 2 \pmod{6}$

Lemma 3.20 $A_3(8, 6, [2, 2]) = 5$, $A_3(14, 6, [2, 2]) \geq 27$.

Proof: For $n = 8$, see Table 1.

For $n = 14$, the required code is constructed on I_{14} with codewords as below.

$$\begin{array}{llllll}
 \langle 0, 2, 5, 9 \rangle & \langle 5, 9, 6, 10 \rangle & \langle 6, 10, 3, 4 \rangle & \langle 0, 7, 11, 12 \rangle & \langle 8, 13, 0, 2 \rangle & \langle 5, 7, 1, 13 \rangle \\
 \langle 7, 9, 2, 4 \rangle & \langle 4, 11, 5, 7 \rangle & \langle 3, 5, 2, 11 \rangle & \langle 2, 4, 10, 13 \rangle & \langle 9, 11, 0, 3 \rangle & \langle 9, 12, 8, 13 \rangle \\
 \langle 1, 8, 3, 5 \rangle & \langle 1, 10, 0, 7 \rangle & \langle 6, 13, 7, 9 \rangle & \langle 1, 13, 4, 11 \rangle & \langle 3, 4, 9, 12 \rangle & \langle 10, 13, 5, 12 \rangle \\
 \langle 0, 4, 1, 8 \rangle & \langle 2, 12, 3, 7 \rangle & \langle 5, 12, 0, 4 \rangle & \langle 8, 10, 9, 11 \rangle & \langle 1, 6, 2, 12 \rangle & \langle 11, 12, 1, 10 \rangle \\
 \langle 3, 7, 8, 10 \rangle & \langle 0, 3, 6, 13 \rangle & \langle 2, 11, 6, 8 \rangle & & &
 \end{array}$$

□

It is easy to see that if there exists a $[2, 2]$ -GDC(6) of type 2^{3t+1} with size $2t(3t+1)$, then there is an optimal $(6t+2, 6, [2, 2])_3$ -code. So we will construct $[2, 2]$ -GDC(6)s of type 2^{3t+1} .

Lemma 3.21 *There exists a $[2, 2]$ -GDC(6) of type 2^{3t+1} with size $2t(3t+1)$ for each $3 \leq t \leq 11$ or $t \in \{14, 17\}$.*

Proof: For $t = 3$, the code is constructed in Lemma 3.7.

For $4 \leq t \leq 11$ or $t \in \{14, 17\}$, let $X_t = \mathbb{Z}_{6t+2}$, and $\mathcal{G}_t = \{\{i, i+3t+1\} : 0 \leq i \leq 3t\}$. Then $(X_t, \mathcal{G}_t, \mathcal{C}_t)$ is the desired $[2, 2]$ -GDC(6) of type 2^{3t+1} , where for $t \neq 5$, \mathcal{C}_t is obtained by developing the elements of \mathbb{Z}_{6t+2} in the codewords listed in Table 3 $+1 \pmod{6t+2}$, and \mathcal{C}_5 is obtained by developing the elements of \mathbb{Z}_{32} in the codewords listed in Table 3 $+2 \pmod{32}$. □

Lemma 3.22 *There exists a $[2, 2]$ -GDC(6) of type 2^{3t+1} with size $2t(3t+1)$ for each $t \geq 12$ and $t \notin \{14, 17\}$.*

Proof: For each $t \geq 12$ and $t \notin P$, take a $[2, 2]$ -GDC(6) of type $24^i 30^j 36^k 42^l 48^m$ with $4i + 5j + 6k + 7l + 8m = t$ from Lemma 3.13. Adjoin two ideal points, and fill in the groups together with the ideal points with $[2, 2]$ -GDC(6)s of type 2^u for $u \in \{13, 16, 19, 22, 25\}$ to obtain the desired GDC.

For each $t \in \{12, 15, 16, 18, 19, 21, 22, 23, 26, 27, 28, 31, 32, 33\}$, take the $[2, 2]$ -GDC(6) constructed in Lemma 3.14. Adjoin two ideal points, and fill in the groups to obtain the desired GDC.

For $t = 13$, take a TD(4, 5) and apply the Fundamental Construction with weight 4 to obtain a $[2, 2]$ -GDC(6) of type 20^4 . Then fill in the groups with $[2, 2]$ -GDC(6)s of type 2^{10} to obtain the required GDC. \square

Summarizing the above results, we have:

Theorem 3.23 $A_3(8, 6, [2, 2]) = 5$, $A_3(14, 6, [2, 2]) \geq 27$, $A_3(6t + 2, 6, [2, 2]) = U(6t + 2, 6, [2, 2])$ for each $t \geq 3$.

3.6 Case of Length $n \equiv 5 \pmod{6}$

Lemma 3.24 $A_3(5, 6, [2, 2]) = 1$, $A_3(11, 6, [2, 2]) = 15$, $A_3(17, 6, [2, 2]) \geq 40$.

Proof: For the first equation, see Table 1. From [44], we have $A_3(11, 6, [2, 2]) \leq A_3(11, 6, 4) = 15$. An optimal $(11, 6, [2, 2])_3$ -code of size 15 is also constructed in [44].

A $(17, 6, [2, 2])_3$ -code with 40 codewords is constructed on $\mathbb{Z}_{16} \cup \{\infty\}$, and is obtained by developing the following base codewords $+4 \pmod{16}$.

$$\begin{array}{cccccc} \langle 1, 5, 8, 14 \rangle & \langle 0, 11, 12, 10 \rangle & \langle 0, 6, 7, 4 \rangle & \langle 3, 10, 15, 6 \rangle & \langle 0, 3, 9, 5 \rangle & \langle 1, 6, \infty, 12 \rangle \\ \langle 1, 2, 15, 9 \rangle & \langle 0, \infty, 13, 15 \rangle & \langle 2, 4, 5, 6 \rangle & \langle 3, 13, 7, 12 \rangle & & \end{array}$$

\square

Lemma 3.25 *There exists a $[2, 2]$ -GDC(6) of type $2^{3t}5^1$ with size $2t(3t+4)$ for each $3 \leq t \leq 8$.*

Proof: Let $X_t = I_{6t+5}$, and $\mathcal{G}_t = \{\{i, i+3t\} : 0 \leq i \leq 3t-1\} \cup \{\{6t, 6t+1, 6t+2, 6t+3, 6t+4\}\}$. Then $(X_t, \mathcal{G}_t, \mathcal{C}_t)$ is a $[2, 2]$ -GDC(6) of type $2^{3t}5^1$, where \mathcal{C}_3 is obtained by developing the codewords in Table 4 under the automorphism group $G = \langle (0 \ 3 \ 6 \ \cdots \ 15)(1 \ 4 \ 7 \ \cdots \ 16)(2 \ 5 \ 8 \ \cdots \ 17)(18 \ 19 \ 20)(21 \ 22) \rangle$, and for $t > 3$, \mathcal{C}_t is obtained by developing the elements of \mathbb{Z}_{6t} in the codewords in Table 4 under the automorphism group $G = \langle (0 \ 3 \ 6 \ \cdots \ 6t-3)(1 \ 4 \ 7 \ \cdots \ 6t-2)(2 \ 5 \ 8 \ \cdots \ 6t-1)(6t)(6t+1)(6t+2)(6t+3)(6t+4) \rangle$. \square

Table 4: Base Codewords of Small $[2, 2]$ -GDC(6)s of Type $2^{3t}5^1$ in Lemma 3.25

t	Codewords							
3	$\langle 1, 5, 3, 22 \rangle$	$\langle 0, 21, 7, 11 \rangle$	$\langle 0, 22, 1, 17 \rangle$	$\langle 1, 20, 2, 15 \rangle$	$\langle 0, 12, 15, 19 \rangle$	$\langle 0, 20, 4, 14 \rangle$	$\langle 2, 8, 5, 19 \rangle$	$\langle 1, 11, 0, 12 \rangle$
4	$\langle 2, 18, 1, 6 \rangle$	$\langle 2, 4, 12, 22 \rangle$	$\langle 1, 13, 16, 19 \rangle$	$\langle 0, 5, 10, 16 \rangle$	$\langle 0, 13, 2, 8 \rangle$	$\langle 2, 25, 18, 22 \rangle$	$\langle 1, 23, 20, 2 \rangle$	$\langle 0, 22, 25, 5 \rangle$
5	$\langle 2, 26, 15, 7 \rangle$	$\langle 1, 9, 17, 27 \rangle$	$\langle 2, 8, 19, 17 \rangle$	$\langle 1, 19, 10, 12 \rangle$	$\langle 0, 4, 1, 7 \rangle$	$\langle 2, 27, 16, 21 \rangle$	$\langle 0, 14, 13, 24 \rangle$	$\langle 1, 28, 0, 14 \rangle$
6	$\langle 2, 18, 21, 9 \rangle$	$\langle 0, 19, 2, 33 \rangle$	$\langle 2, 26, 0, 4 \rangle$	$\langle 1, 6, 31, 11 \rangle$	$\langle 1, 15, 34, 8 \rangle$	$\langle 2, 32, 19, 24 \rangle$	$\langle 2, 31, 15, 16 \rangle$	$\langle 2, 33, 25, 27 \rangle$
7	$\langle 0, 13, 11, 32 \rangle$	$\langle 1, 27, 24, 4 \rangle$	$\langle 0, 10, 26, 30 \rangle$	$\langle 0, 1, 9, 22 \rangle$	$\langle 1, 19, 23, 28 \rangle$	$\langle 2, 34, 22, 3 \rangle$	$\langle 0, 12, 20, 6 \rangle$	$\langle 2, 20, 29, 23 \rangle$
8	$\langle 1, 3, 2, 20 \rangle$	$\langle 2, 30, 28, 12 \rangle$	$\langle 1, 26, 7, 25 \rangle$	$\langle 2, 23, 4, 28 \rangle$	$\langle 0, 28, 23, 37 \rangle$	$\langle 2, 26, 22, 0 \rangle$	$\langle 2, 37, 24, 13 \rangle$	$\langle 1, 0, 4, 30 \rangle$
9	$\langle 2, 38, 15, 31 \rangle$	$\langle 0, 31, 27, 2 \rangle$	$\langle 0, 11, 15, 12 \rangle$	$\langle 1, 21, 39, 26 \rangle$	$\langle 1, 10, 2, 23 \rangle$	$\langle 2, 36, 33, 25 \rangle$	$\langle 1, 12, 11, 5 \rangle$	$\langle 2, 40, 16, 21 \rangle$
10	$\langle 2, 8, 35, 11 \rangle$	$\langle 2, 39, 9, 10 \rangle$	$\langle 0, 8, 24, 7 \rangle$	$\langle 1, 25, 31, 22 \rangle$	$\langle 0, 3, 17, 9 \rangle$	$\langle 0, 10, 26, 40 \rangle$	$\langle 2, 15, 38, 0 \rangle$	$\langle 0, 24, 28, 22 \rangle$
11	$\langle 2, 15, 38, 0 \rangle$	$\langle 0, 24, 28, 22 \rangle$	$\langle 2, 29, 35, 19 \rangle$	$\langle 2, 36, 9, 41 \rangle$	$\langle 2, 33, 11, 26 \rangle$	$\langle 1, 16, 13, 4 \rangle$	$\langle 1, 10, 21, 32 \rangle$	$\langle 2, 4, 21, 39 \rangle$
12	$\langle 2, 18, 31, 43 \rangle$	$\langle 1, 29, 5, 30 \rangle$	$\langle 2, 30, 42, 22 \rangle$	$\langle 1, 7, 20, 8 \rangle$	$\langle 2, 12, 45, 1 \rangle$	$\langle 0, 30, 7, 25 \rangle$	$\langle 1, 46, 38, 24 \rangle$	$\langle 2, 32, 13, 40 \rangle$
13	$\langle 1, 45, 0, 26 \rangle$	$\langle 1, 42, 6, 35 \rangle$	$\langle 1, 43, 11, 3 \rangle$	$\langle 2, 27, 46, 37 \rangle$	$\langle 0, 44, 16, 2 \rangle$	$\langle 1, 17, 39, 44 \rangle$	$\langle 1, 25, 9, 15 \rangle$	$\langle 0, 38, 1, 41 \rangle$
14	$\langle 0, 36, 33, 3 \rangle$	$\langle 0, 45, 18, 30 \rangle$	$\langle 2, 50, 42, 25 \rangle$	$\langle 2, 52, 40, 27 \rangle$	$\langle 1, 3, 45, 7 \rangle$	$\langle 0, 1, 15, 16 \rangle$	$\langle 1, 4, 34, 22 \rangle$	$\langle 2, 8, 19, 39 \rangle$
15	$\langle 0, 19, 47, 41 \rangle$	$\langle 2, 32, 34, 12 \rangle$	$\langle 0, 43, 32, 27 \rangle$	$\langle 2, 29, 36, 28 \rangle$	$\langle 0, 22, 48, 17 \rangle$	$\langle 0, 7, 34, 20 \rangle$	$\langle 2, 5, 38, 17 \rangle$	$\langle 0, 5, 9, 6 \rangle$
16	$\langle 0, 5, 9, 6 \rangle$	$\langle 1, 9, 51, 32 \rangle$	$\langle 2, 48, 0, 43 \rangle$	$\langle 0, 36, 44, 14 \rangle$	$\langle 2, 41, 22, 15 \rangle$	$\langle 1, 21, 10, 12 \rangle$	$\langle 2, 51, 21, 46 \rangle$	$\langle 16, 39, 26, 50 \rangle$
17	$\langle 16, 39, 26, 50 \rangle$	$\langle 16, 4, 23, 20 \rangle$	$\langle 16, 2, 10, 7 \rangle$	$\langle 2, 49, 18, 37 \rangle$	$\langle 0, 13, 38, 49 \rangle$	$\langle 3, 13, 5, 14 \rangle$	$\langle 0, 31, 29, 52 \rangle$	

Lemma 3.26 *There exists a $[2, 2]$ -GDC(6) of type $2^{3t}5^1$ with size $2t(3t + 4)$ for each $t \geq 12$ and $t \notin \{13, 14, 17\}$.*

Proof: For each $t \geq 12$ and $t \notin P$, take a $[2, 2]$ -GDC(6) of type $24^i 30^j 36^k 42^l 48^m$ with $4i + 5j + 6k + 7l + 8m = t$ from Lemma 3.13. Adjoin five ideal points, and fill in the groups together with the ideal points with $[2, 2]$ -GDC(6)s of type $2^{3s}5^1$ for $s \in \{4, 5, 6, 7, 8\}$ to obtain the desired GDC.

For each $t \in \{12, 15, 16, 18, 19, 21, 22, 23, 26, 27, 28, 31, 32, 33\}$, take the $[2, 2]$ -GDC(6) constructed in Lemma 3.14. Adjoin five ideal points, and fill in the groups together with the ideal points with $[2, 2]$ -GDC(6)s of type $2^{3s}5^1$ for $s \in \{3, 4, 5, 6, 7\}$ to obtain the desired GDC. \square

Lemma 3.27 $A_3(6t+5, 6, [2, 2]) \geq U(6t+5, 6, [2, 2]) - 1$ for each $t \geq 3$ and $t \notin \{9, 10, 11, 14\}$; $A_3(6t + 5, 6, [2, 2]) \geq U(6t + 5, 6, [2, 2]) - 2$ for $t = 14$.

Proof: For each $t \geq 3$ and $t \notin \{9, 10, 11, 13, 14, 17\}$, take a $[2, 2]$ -GDC(6) of type $2^{3t}5^1$ from Lemmas 3.25 and 3.26, and fill in the group of size 5 with an optimal $(5, 6, [2, 2])_3$ -code with one codeword to get the desired code.

For each $t \in \{13, 14, 17\}$, take a $[2, 2]$ -GDC(6) of type $g^u m^1$ with $(g, u, m) \in \{(18, 4, 6), (18, 4, 12), (24, 4, 6)\}$ from Lemma 3.12. Adjoin 5 ideal points, fill in the groups of size g together with the ideal points with $[2, 2]$ -GDC(6)s of type $2^{g/2}5^1$, and fill in the group of size m together with the ideal points with a code of length 11 or length 17 from Lemma 3.24. \square

Lemma 3.28 $A_3(24t + 5, 6, [2, 2]) = U(24t + 5, 6, [2, 2])$ for each $t \geq 3$.

Proof: Take a $[2, 2]$ -GDC(6) of type $(6t + 1)^4$ with size $2(6t + 1)^2$ from Lemma 3.6. Adjoin one extra point and fill in the groups with optimal codes of length $6t + 2$ from Theorem 3.23 to obtain the desired code. \square

Lemma 3.29 $A_3(6t + 5, 6, [2, 2]) = U(6t + 5, 6, [2, 2])$ for each $t \geq 130$.

Proof: Take a TD(7, m) with $m \geq 23$ and $m \notin \{26, 30, 34, 38, 46, 60\}$ from Theorem 2.7 and apply the Fundamental Construction, assigning weight 6 to all points in the first five groups and weights 0 or 6 to the points in the last two groups. Note that there exist $[2, 2]$ -GDC(6)s of type 6^s for $s \in \{5, 6, 7\}$ by Lemma 3.8. The result is a $[2, 2]$ -GDC(6) of type $(6m)^5(6x)^172^1$ with $x \in [3, 8] \cup [18, 23]$. Adjoin five ideal points. Fill in the first six groups together with the ideal points with $[2, 2]$ -GDC(6)s of types $2^{3m}5^1$ and $2^{3x}5^1$, and fill in the group of size 72 together with the five ideal points with an optimal $(77, 6, [2, 2])_3$ -code from Lemma 3.28. The result is an optimal code of length $6t + 5$ with $t = 5m + x + 12$, as desired. \square

Lemma 3.30 $A_3(6t + 5, 6, [2, 2]) = U(6t + 5, 6, [2, 2])$ for each $t \in \{54, 55\}$ or $63 \leq t \leq 129$.

Proof: For each $t \in \{54, 55\}$, take a TD(5, 12) from Theorem 2.7 and apply the Fundamental Construction, assigning weight 6 to all points in the first four groups and weights 3 or 6 to points in the last group. The result is a $[2, 2]$ -GDC(6) of type 72^436^1 or type 72^442^1 . Then adjoin five ideal points and fill in the groups with $[2, 2]$ -GDC(6)s of type 2^u5^1 for $u \in \{18, 21, 36\}$ and an optimal $(77, 6, [2, 2])_3$ -code to obtain the desired code.

For each $63 \leq t \leq 76$, take a TD(9, 8) from Theorem 2.7 and apply the Fundamental Construction, assigning weight 6 to all points in the first six groups, weight 9 to the points in the last group and weights 0 or 6 to the remaining points. Note that there exist $[2, 2]$ -GDC(6)s of type 6^s9^1 for $s \in \{6, 7, 8\}$ by Lemma 3.11. The result is a $[2, 2]$ -GDC(6) of type $48^6(6x)^1(6y)^172^1$ with $x, y \in \{0, 3, 4, 5, 6, 7, 8\}$. Adjoin five ideal points and fill in the groups to obtain an optimal code of length $6t + 5$ with $t = 60 + x + y$.

For each $77 \leq t \leq 100$ and $t \notin \{85, 86\}$, take a TD(9, u) with $u \in \{9, 11\}$ from Theorem 2.7 and remove one point to redefine the groups to obtain a $\{9, u\}$ -GDD of type $8^u(u - 1)^1$. Then apply the Fundamental Construction, assigning weight 6 to the points in the first $u - 2$ groups of size 8, weights 0 or 9 to the points in the group of size $u - 1$ and weights 0 or 6 to the remaining points. Note that there exist $[2, 2]$ -GDC(6)s of types 6^s9^1 with $s \in \{6, 7, 8\}$ and 6^s with $5 \leq s \leq 11$ by Lemmas 3.8 and 3.11. The result is a $[2, 2]$ -GDC(6) of type $48^{u-2}(6x)^1(6y)^172^1$ with $x, y \in \{0, 3, 4, 5, 6, 7, 8\}$. Adjoin five ideal points and fill in the groups to obtain an optimal code of length $6t + 5$ with $t = 8u + x + y - 4$.

For each $t \in \{85, 86\}$, take a TD(8, 11) from Theorem 2.7 and remove one point to redefine the groups to obtain an $\{8, 11\}$ -GDD of type $7^{11}10^1$. Then apply the Fundamental Construction, assigning weight 6 to the points in the first ten groups of size 7, weights 0 or 9 to the points in the group of size 10 and weights 0 or 6 to the remaining points. The result

Table 5: Base Codewords of Small Optimal $(6t + 4, 6, [2, 2])_3$ -Codes in Lemma 3.32

t	Codewords								
4	$\langle 0, 15, 21, 24 \rangle$	$\langle 0, 6, 9, 11 \rangle$	$\langle 1, 22, 26, 21 \rangle$	$\langle 1, 3, 11, 15 \rangle$	$\langle 0, 1, 18, 19 \rangle$	$\langle 1, 6, 23, 16 \rangle$	$\langle 1, 5, 24, 12 \rangle$	$\langle 1, 4, 17, 2 \rangle$	$\langle 0, 12, 14, 20 \rangle$
5	$\langle 0, 11, 17, 21 \rangle$	$\langle 1, 19, 10, 18 \rangle$	$\langle 1, 9, 12, 23 \rangle$	$\langle 1, 33, 30, 20 \rangle$	$\langle 0, 7, 14, 20 \rangle$	$\langle 0, 24, 23, 15 \rangle$	$\langle 0, 8, 1, 3 \rangle$	$\langle 0, 19, 31, 13 \rangle$	
	$\langle 1, 31, 2, 21 \rangle$	$\langle 0, 28, 12, 16 \rangle$	$\langle 0, 4, 2, 9 \rangle$						
6	$\langle 0, 19, 10, 12 \rangle$	$\langle 1, 2, 13, 31 \rangle$	$\langle 1, 26, 0, 3 \rangle$	$\langle 0, 25, 5, 8 \rangle$	$\langle 0, 4, 32, 13 \rangle$	$\langle 0, 6, 22, 7 \rangle$	$\langle 1, 19, 11, 17 \rangle$	$\langle 0, 3, 27, 31 \rangle$	
	$\langle 1, 5, 18, 12 \rangle$	$\langle 1, 20, 6, 10 \rangle$	$\langle 0, 2, 20, 35 \rangle$	$\langle 1, 4, 27, 28 \rangle$	$\langle 1, 7, 15, 36 \rangle$				
7	$\langle 1, 35, 40, 24 \rangle$	$\langle 1, 31, 0, 29 \rangle$	$\langle 1, 9, 10, 12 \rangle$	$\langle 0, 16, 43, 21 \rangle$	$\langle 0, 22, 17, 7 \rangle$	$\langle 0, 32, 9, 25 \rangle$	$\langle 1, 33, 14, 20 \rangle$	$\langle 0, 2, 13, 1 \rangle$	
	$\langle 1, 18, 7, 21 \rangle$	$\langle 1, 43, 26, 38 \rangle$	$\langle 1, 23, 3, 41 \rangle$	$\langle 0, 34, 26, 8 \rangle$	$\langle 0, 18, 37, 33 \rangle$	$\langle 0, 40, 4, 36 \rangle$	$\langle 1, 11, 8, 32 \rangle$		
8	$\langle 0, 50, 3, 11 \rangle$	$\langle 0, 49, 38, 21 \rangle$	$\langle 1, 33, 37, 43 \rangle$	$\langle 1, 18, 40, 38 \rangle$	$\langle 1, 9, 30, 2 \rangle$	$\langle 1, 12, 13, 27 \rangle$	$\langle 1, 24, 32, 49 \rangle$	$\langle 0, 4, 18, 47 \rangle$	
	$\langle 0, 17, 12, 39 \rangle$	$\langle 1, 39, 3, 14 \rangle$	$\langle 1, 20, 51, 19 \rangle$	$\langle 1, 34, 41, 44 \rangle$	$\langle 1, 7, 35, 16 \rangle$	$\langle 0, 45, 42, 23 \rangle$	$\langle 0, 24, 16, 9 \rangle$	$\langle 0, 27, 26, 32 \rangle$	
	$\langle 0, 6, 36, 40 \rangle$								
9	$\langle 1, 31, 49, 55 \rangle$	$\langle 0, 1, 17, 39 \rangle$	$\langle 1, 14, 41, 46 \rangle$	$\langle 1, 18, 36, 11 \rangle$	$\langle 0, 31, 52, 15 \rangle$	$\langle 1, 6, 35, 4 \rangle$	$\langle 0, 21, 4, 28 \rangle$	$\langle 0, 5, 37, 9 \rangle$	
	$\langle 0, 10, 13, 22 \rangle$	$\langle 1, 26, 51, 52 \rangle$	$\langle 1, 2, 40, 45 \rangle$	$\langle 1, 10, 24, 16 \rangle$	$\langle 1, 23, 21, 34 \rangle$	$\langle 0, 16, 23, 35 \rangle$	$\langle 1, 48, 20, 30 \rangle$	$\langle 0, 8, 54, 55 \rangle$	
	$\langle 1, 47, 15, 32 \rangle$	$\langle 1, 7, 56, 9 \rangle$	$\langle 0, 34, 20, 36 \rangle$						

is a $[2, 2]$ -GDC(6) of type $42^{10}(6x)^1 72^1$ with $x \in \{3, 4\}$. Adjoin five ideal points and fill in the groups to obtain the desired optimal code.

Finally, for $101 \leq t \leq 129$, take a TD(11, 16) and apply the Fundamental Construction, assigning weight 6 to the points in the first five groups and weights 0 or 6 to the remaining points. The result is a $[2, 2]$ -GDC(6) of type $96^5(6x_1)^1(6x_2)^1 \dots (6x_5)^1 72^1$ with $x_1, x_2, \dots, x_5 \in \{0, 3, 4, 5, 6, 7, 8\}$. Adjoin five ideal points and fill in the groups to complete the proof. \square

Summarizing the above results, we have:

Theorem 3.31 *Let $Q^{(5)} = \{9, 10, 11\}$, $Q_1^{(5)} = \{3, 4, 5, 6, 7, 8, 13, 57, 58, 59, 61, 62\} \cup \{t : 15 \leq t \leq 53, t \not\equiv 0 \pmod{4}\}$, and $Q_2^{(5)} = \{2, 14\}$. Then $A_3(5, 6, [2, 2]) = 1$, $A_3(11, 6, [2, 2]) = 15$, $A_3(6t + 5, 6, [2, 2]) = U(6t + 5, 6, [2, 2])$ for each $t \geq 2$ and $t \notin Q^{(5)} \cup Q_1^{(5)} \cup Q_2^{(5)}$. Furthermore, we have*

1. $U(6t + 5, 6, [2, 2]) - 1 \leq A_3(6t + 5, 6, [2, 2]) \leq U(6t + 5, 6, [2, 2])$ for each $t \in Q_1^{(5)}$;
2. $U(6t + 5, 6, [2, 2]) - 2 \leq A_3(6t + 5, 6, [2, 2]) \leq U(6t + 5, 6, [2, 2])$ for each $t \in Q_2^{(5)}$.

3.7 Case of Length $n \equiv 4 \pmod{6}$

Lemma 3.32 $A_3(6t + 4, 6, [2, 2]) = U(6t + 4, 6, [2, 2])$ for $t = 1$ or $4 \leq t \leq 9$.

Proof: For $t = 1$, see Table 1. For $4 \leq t \leq 9$, let $X_t = \mathbb{Z}_{6t+4}$. Then (X_t, \mathcal{C}_t) is the desired optimal $(6t + 4, 6, [2, 2])_3$ -code, where \mathcal{C}_t is obtained by developing the elements of \mathbb{Z}_{6t+4} in the codewords listed in Table 5 +2 (mod $6t + 4$). \square

Lemma 3.33 $A(6t + 4, 6, [2, 2]) = U(6t + 4, 6, [2, 2])$ for $t \geq 142$.

Proof: Take a $\text{TD}(8, m)$ with $m \geq 23$ and $m \notin \{26, 28, 30, 33, 34, 35, 38, 39, 42, 44, 46, 51, 52, 54, 58, 60, 62, 66, 68, 74\}$ from Theorem 2.7. Apply the Fundamental Construction with weight 6 to all points in the first 6 groups, x points in the seventh group, and weight 3 to 3 points in the last group. The other points are given weight 0. Noting that there exist $[2, 2]$ -GDC(6)s of type 6^s for $s \in \{6, 7\}$ by Lemma 3.8, and $[2, 2]$ -GDC(6)s of type $6^s 3^1$ for $s \in \{6, 7\}$ by Lemma 3.10. The result is a $[2, 2]$ -GDC(6) of type $(6m)^6(6x)^1 9^1$ for $x = 0$ or $3 \leq x \leq m$. Adjoin one ideal point. Fill in the first seven groups together with the ideal point with optimal codes of lengths $6m + 1$ and $6x + 1$ from Theorem 3.18, and fill in the group of size 9 together with the ideal point with an optimal $(10, 6, [2, 2])_3$ -code. The result is an optimal code of length $6t + 4$ with $t = 6m + x + 1$. That includes all integers $t \geq 142$. \square

Lemma 3.34 $A(6t + 4, 6, [2, 2]) = U(6t + 4, 6, [2, 2])$ for $t = 43$ or $46 \leq t \leq 141$ and $t \neq 51$.

Proof: Take a $\text{TD}(k, m)$ from Theorem 2.7 with $m \in \{7, 8, 9, 13\}$ and $8 \leq k \leq 12$, $k \leq m + 1$. Apply the Fundamental Construction with weight 6 to all the points in the first 6 groups, weight 3 to 3 points in the last group, and weight 6 to x_i points in the remaining $k - 7$ groups for $1 \leq i \leq k - 7$. The remaining points are given weight 0. Take $x_i = 0$ or $3 \leq x_i \leq m$. Noting that there exist $[2, 2]$ -GDC(6)s of type 6^s for $6 \leq s \leq 11$ by Lemma 3.8, and $[2, 2]$ -GDC(6)s of type $6^s 3^1$ for $6 \leq s \leq 11$ by Lemma 3.10. The result is a $[2, 2]$ -GDC(6) of type $(6m)^6(6x_1)^1 \dots (6x_{k-7})^1 9^1$. Adjoin one ideal point. Fill in the first $k - 1$ groups together with the ideal point with optimal codes of lengths $6m + 1$ and $6x_i + 1$ from Theorem 3.18, and fill in the group of size 9 together with the ideal point with an optimal $(10, 6, [2, 2])_3$ -code. The result is an optimal code of length $6t + 4$ with $t = 6m + \sum_{i=1}^{k-7} x_i + 1$.

Taking $k = 8$ and $m = 7$, we obtain $t \in \{43\} \cup [46, 50]$. Taking $k = 9$ and $m = 8$, we obtain $t \in [52, 65]$. Taking $k = 10$ and $m = 9$, we obtain an optimal code of length $t \in [66, 82]$. Taking $k = 12$ and $m = 13$, we obtain $t \in [83, 141]$. \square

Lemma 3.35 $A(6t + 4, 6, [2, 2]) = U(6t + 4, 6, [2, 2])$ for $t \in \{24\} \cup [32, 34] \cup [36, 42] \cup \{44\}$.

Proof: Take a $\text{TD}(5, m)$ with $m \in \{5, 7, 8, 9\}$ from Theorem 2.7. Apply the Fundamental Construction with weight 6 to all the points in the first 4 groups, x points in the last group, and weight 3 to y points in the last group, such that $x + y = m$. Noting that there exists a $[2, 2]$ -GDC(6) of type 6^5 by Lemma 3.8, and a $[2, 2]$ -GDC(6) of type $6^4 3^1$ by Lemma 3.10. The result is a $[2, 2]$ -GDC(6) of type $(6m)^4(6x + 3y)^1$. Adjoin one ideal point. Fill in the first 4 groups together with the ideal point with optimal codes of length $6m + 1$ from Theorem 3.18, and fill in the group of size $6x + 3y$ together with the ideal point with an optimal $(6x + 3y + 1, 6, [2, 2])_3$ -code from Lemma 3.32. The result is an optimal code of length $6t + 4$ with $t = 4m + x + \frac{y-1}{2}$. For each desired t , the parameters (m, x, y) and the code of length $s = 6x + 3y + 1$ to be filled in are given in Table 6. \square

Lemma 3.36 $A(6t + 4, 6, [2, 2]) = U(6t + 4, 6, [2, 2])$ for each $t \in \{10, 11, 13, 16, 17, 18, 19, 21, 22, 23, 25, 26, 28, 31, 45, 51\}$.

Table 6: The Parameters for Lemma 3.35

t	(m, x, y)	s	t	(m, x, y)	s	t	(m, x, y)	s
24	(5, 4, 1)	28	32	(7, 2, 5)	28	33	(7, 4, 3)	34
34	(7, 6, 1)	40	36	(8, 1, 7)	28	37	(8, 3, 5)	34
38	(8, 5, 3)	40	39	(8, 7, 1)	46	40	(9, 0, 9)	28
41	(9, 2, 7)	34	42	(9, 4, 5)	40	44	(9, 8, 1)	52

Table 7: The Parameters for Lemma 3.36

t	n	$g^u m^1 \times w$	Source	a	S
10	64	$3^7 \times 3$	Lemma 3.9	1	10
13	82	$6^4 3^1 \times 3$	Lemma 3.10	1	19, 10
16	100	$3^{11} \times 3$	Lemma 3.9	1	10
18	112	$4^4 \times 7$	Lemma 3.6	0	28
19	118	$3^{13} \times 3$	Lemma 3.9	1	10
21	130	$2^{13} \times 5$	Lemma 3.21	0	10
22	136	$6^7 3^1 \times 3$	Lemma 3.10	1	19, 10
25	154	$6^7 9^1 \times 3$	Lemma 3.11	1	19, 28
26	160	$4^4 \times 10$	Lemma 3.6	0	40
28	172	$6^8 9^1 \times 3$	Lemma 3.11	1	19, 28
31	190	$3^7 \times 9$	Lemma 3.9	1	28
45	274	$3^7 \times 13$	Lemma 3.9	1	40
51	310	$1^{31} \times 10$	Lemma 3.16	0	10

Proof: For $t = 11$, take a $[2, 2]$ -GDC(6) of type 10^7 from Lemma 3.9. Fill in the groups with optimal $(10, 6, [2, 2])_3$ -codes to obtain the desired code.

For $t = 17$, take a $[2, 2]$ -GDC(6) of type $24^4 9^1$ from Lemma 3.12. Adjoin an ideal point, and fill in each group together with the ideal point with an optimal $(25, 6, [2, 2])_3$ -code or an optimal $(10, 6, [2, 2])_3$ -code to obtain the desired code. For $t = 23$, we proceed similarly, starting instead with a $[2, 2]$ -GDC(6) of type $18^6 33^1$ from Lemma 3.12.

For each desired $t \in \{10, 13, 16, 18, 19, 21, 22, 25, 26, 28, 31, 45, 51\}$, we list the parameters to obtain the optimal code of length $6t + 4$ in Table 7. We inflate a $[2, 2]$ -GDC(6) of type $g^u m^1$ (from “Source”) with weight w , adjoin a ideal point with $a = 0$ or 1 , and fill in the groups together with the ideal point with optimal codes of length $s \in S$ to obtain the desired code. \square

Lemma 3.37 $A(6t + 4, 6, [2, 2]) = U(6t + 4, 6, [2, 2])$ for $t \in \{29, 35\}$.

Proof: For $t = 29$, take a $[2, 2]$ -GDC(6) of type 6^7 from Lemma 3.8; apply the Inflation Construction with weight 4, using 4-MGDDs of type 4^4 (see [22]) as input designs, to obtain a 4-DGDD of type $(24, 6^4)^7$ with the CCC property. Adjoin 9 ideal points, and fill in $[2, 2]$ -GDC(6)s of type $6^7 9^1$ to obtain a $[2, 2]$ -GDC(6) of type $24^7 9^1$. Adjoin one more ideal point,

and fill in the groups together with the ideal point with optimal codes of lengths 25 and 10 to obtain the desired code.

For $t = 35$, take a $[2, 2]$ -GDC(6) of type 6^7 and remove all the points in the last group; apply the Inflation Construction with weight 5, using 4-MGDDs of type 5^4 (see [22]) and resolvable 3-MGDDs of type 5^3 (see [38]) as input designs, to obtain a $\{3, 4\}$ -DGDD of type $(30, 6^5)^6$ with the CCC property, whose triples fall into 48 parallel classes. Adjoin 24 infinite points to complete the parallel classes, and then adjoin further 9 ideal points, fill in a $[2, 2]$ -GDC(6) of type $6^6 9^1$ to obtain a $[2, 2]$ -GDC(6) of type $30^6 33^1$. Adjoin one more point, and fill in the groups together with the ideal point with optimal codes of lengths 31 and 34 to obtain the desired code. \square

Lemma 3.38 *There exists a $[2, 2]$ -GDC(6) of type $1^{27} 4^1$ with size 153.*

Proof: Let $X = (\mathbb{Z}_9 \times \{0, 1, 2\}) \cup (\mathbb{Z}_3 \times \{3\}) \cup \{\infty\}$. The point set $(\mathbb{Z}_3 \times \{3\}) \cup \{\infty\}$ forms the group of size 4. Define $\alpha : X \rightarrow X$ as $x_y \rightarrow (x + 1)_y$ where the addition is modulo 9 if $y \in \{0, 1, 2\}$, and modulo 3 if $y \in \{3\}$. The point ∞ is fixed by α . Develop the following 17 base codewords with α :

$$\begin{array}{llllll} \langle 7_2, 0_3, 3_2, 5_0 \rangle & \langle 5_1, 0_3, 1_0, 3_0 \rangle & \langle 8_2, 0_3, 6_1, 4_1 \rangle & \langle 5_0, 0_2, 0_3, 8_2 \rangle & \langle 1_1, 0_1, 0_3, 5_1 \rangle & \langle 3_0, 7_0, 0_3, 4_2 \rangle \\ \langle 5_1, \infty, 0_0, 4_2 \rangle & \langle 0_0, 7_2, \infty, 0_1 \rangle & \langle 5_1, 7_1, 7_0, 6_0 \rangle & \langle 8_1, 0_2, 1_2, 4_2 \rangle & \langle 7_1, 4_1, 2_2, 4_2 \rangle & \langle 1_0, 7_0, 6_2, 3_1 \rangle \\ \langle 0_2, 6_1, 3_0, 0_0 \rangle & \langle 1_2, 8_2, 5_0, 0_0 \rangle & \langle 5_2, 8_2, 0_1, 8_1 \rangle & \langle 4_0, 3_0, 1_1, 7_1 \rangle & \langle 0_0, 2_0, 1_1, 2_2 \rangle & \end{array}$$

\square

Lemma 3.39 $A(6t + 4, 6, [2, 2]) = U(6t + 4, 6, [2, 2])$ for $t \in \{27, 30\}$.

Proof: For each $t \in \{27, 30\}$, take a $[2, 2]$ -GDC(6) of type 9^6 or type $9^5 15^1$ from Lemma 3.12. Apply the Inflation Construction with weight 3 to obtain a $[2, 2]$ -GDC(6) of type 27^6 or type $27^5 45^1$. Then adjoin four ideal points and fill in the groups together with these ideal points with $[2, 2]$ -GDC(6)s of type $1^{27} 4^1$ and an optimal code of length 31 or length 49 to obtain the desired code. \square

Combing the above lemmas, we have the following result:

Theorem 3.40 $A_3(6t+4, 6, [2, 2]) = U(6t+4, 6, [2, 2])$ for each $t \geq 1$, $t \notin \{2, 3, 12, 14, 15, 20\}$.

3.8 Case of Length $n \equiv 3 \pmod{6}$

Theorem 3.41 $A_3(6t + 3, 6, [2, 2]) = U(6t + 3, 3)$ for each $t \geq 1$.

Proof: For each $t \geq 1$, $t \notin \{2, 3, 12, 14, 15, 20\}$, remove one point and the related codewords from an optimal $(6t + 4, 6, [2, 2])_3$ -code from Theorem 3.40 to obtain the desired code.

For $t \in \{2, 14, 20\}$, let $X_t = \mathbb{Z}_{6t+3}$. Then (X_t, \mathcal{C}_t) is the desired optimal $(6t + 3, 6, [2, 2])_3$ -code, where \mathcal{C}_t is obtained by developing the elements of \mathbb{Z}_{6t+3} in the following codewords $+1 \pmod{6t+3}$.

$t = 2$: $\langle 0, 3, 4, 14 \rangle \langle 0, 5, 7, 13 \rangle$

$t = 14$:

$\langle 0, 24, 47, 61 \rangle \langle 0, 70, 74, 9 \rangle \langle 0, 41, 16, 36 \rangle \langle 0, 22, 25, 53 \rangle \langle 0, 28, 83, 66 \rangle \langle 0, 39, 52, 50 \rangle$
 $\langle 0, 57, 51, 75 \rangle \langle 0, 14, 49, 5 \rangle \langle 0, 58, 68, 27 \rangle \langle 0, 72, 6, 45 \rangle \langle 0, 20, 54, 84 \rangle \langle 0, 43, 33, 32 \rangle$
 $\langle 0, 85, 69, 40 \rangle \langle 0, 1, 80, 8 \rangle$

$t = 20$:

$\langle 0, 16, 46, 63 \rangle \langle 0, 34, 122, 9 \rangle \langle 0, 10, 43, 74 \rangle \langle 0, 92, 85, 7 \rangle \langle 0, 27, 71, 87 \rangle \langle 0, 17, 2, 8 \rangle$
 $\langle 0, 5, 109, 82 \rangle \langle 0, 23, 37, 55 \rangle \langle 0, 24, 22, 94 \rangle \langle 0, 6, 21, 79 \rangle \langle 0, 72, 35, 69 \rangle \langle 0, 58, 1, 84 \rangle$
 $\langle 0, 18, 54, 59 \rangle \langle 0, 11, 102, 3 \rangle \langle 0, 45, 93, 97 \rangle \langle 0, 4, 57, 80 \rangle \langle 0, 13, 42, 62 \rangle \langle 0, 56, 81, 68 \rangle$
 $\langle 0, 20, 95, 39 \rangle \langle 0, 83, 50, 61 \rangle$

For $t = 3$, the $[2, 2]$ -GDC(6) of type 3^7 constructed in Lemma 3.9 is the desired code.

For $t \in \{12, 15\}$, take a $[2, 2]$ -GDC(6) of type 18^4 (see Lemma 3.9) or type 18^5 . Adjoin 3 ideal points, and fill in the groups together with these ideal points with $[2, 2]$ -GDC(6)s of type 3^7 to obtain the desired code. \square

4 Determining the Value of $A_3(n, 6, [3, 1])$

In this section, we focus on the determination for the exact values of $A_3(n, 6, [3, 1])$ for all positive integers n .

4.1 Some $[3, 1]$ -GDC(6)s

Lemma 4.1 *There is a $[3, 1]$ -GDC(6) of type 3^{3t+1} with size $3t(3t + 1)$ for each $t \in \{2, 6\}$.*

Proof: Let $X_t = \mathbb{Z}_{9t+3}$, and $\mathcal{G}_t = \{\{0, 3t + 1, 6t + 2\} + i : 0 \leq i \leq 3t\}$. Then $(X_t, \mathcal{G}_t, \mathcal{C}_t)$ is a $[3, 1]$ -GDC(6) of type 3^{3t+1} with size $3t(3t + 1)$, where \mathcal{C}_2 is obtained by developing the elements of \mathbb{Z}_{21} in the following codewords $+3 \pmod{21}$, and \mathcal{C}_6 is obtained by developing the elements of \mathbb{Z}_{57} in the following codewords $+1 \pmod{57}$.

$t = 2$: $\langle 0, 1, 13, 18 \rangle \langle 1, 4, 14, 5 \rangle \langle 0, 10, 12, 16 \rangle \langle 2, 5, 7, 1 \rangle \langle 0, 2, 8, 3 \rangle \langle 0, 6, 17, 5 \rangle$

$t = 6$: $\langle 0, 9, 7, 36 \rangle \langle 0, 1, 6, 21 \rangle \langle 0, 26, 8, 30 \rangle \langle 0, 3, 44, 40 \rangle \langle 0, 11, 43, 28 \rangle \langle 0, 10, 33, 45 \rangle$ \square

Table 8: Base Codewords of Small $[3, 1]$ -GDC(6)s of type 6^{3t+1} in Lemma 4.2

t	Codewords
2	$\langle 0, 38, 32, 1 \rangle \langle 0, 12, 34, 29 \rangle \langle 0, 23, 26, 25 \rangle \langle 0, 9, 27, 40 \rangle$
3	$\langle 0, 47, 43, 15 \rangle \langle 0, 21, 23, 35 \rangle \langle 0, 22, 55, 41 \rangle \langle 0, 34, 31, 25 \rangle \langle 0, 36, 44, 45 \rangle \langle 0, 7, 18, 6 \rangle$
4	$\langle 0, 19, 47, 33 \rangle \langle 0, 49, 69, 7 \rangle \langle 0, 34, 40, 10 \rangle \langle 0, 46, 11, 23 \rangle \langle 0, 1, 74, 62 \rangle \langle 0, 2, 53, 70 \rangle$ $\langle 0, 37, 22, 30 \rangle \langle 0, 3, 21, 45 \rangle$
5	$\langle 0, 54, 1, 87 \rangle \langle 0, 71, 12, 63 \rangle \langle 0, 7, 34, 65 \rangle \langle 0, 76, 24, 29 \rangle \langle 0, 92, 19, 41 \rangle \langle 0, 11, 14, 81 \rangle$ $\langle 0, 57, 36, 83 \rangle \langle 0, 66, 94, 8 \rangle \langle 0, 35, 18, 13 \rangle \langle 0, 6, 56, 15 \rangle$

Lemma 4.2 *There exists a $[3, 1]$ -GDC(6) of type 6^{3t+1} with size $12t(3t+1)$ for each $2 \leq t \leq 5$.*

Proof: Let $X_t = \mathbb{Z}_{18t+6}$, and $\mathcal{G}_t = \{\{0, 3t+1, 6t+2, \dots, 15t+5\} + i : 0 \leq i \leq 3t\}$. Then $(X_t, \mathcal{G}_t, \mathcal{C}_t)$ is a $[3, 1]$ -GDC(6) of type 6^{3t+1} , where \mathcal{C}_t is obtained by developing the elements of \mathbb{Z}_{18t+6} in the codewords in Table 8 $+1 \pmod{18t+6}$. \square

Lemma 4.3 *There is a $[3, 1]$ -GDC(6) of type 9^t with size $9t(t-1)$ for each $t \in \{4, 5, 6, 8, 9, 11, 12, 14, 15, 18, 23\}$.*

Proof: Let $X_t = \mathbb{Z}_{9t}$, and $\mathcal{G}_t = \{\{0, t, 2t, \dots, 8t\} + i : 0 \leq i \leq t-1\}$. Then $(X_t, \mathcal{G}_t, \mathcal{C}_t)$ is a $[3, 1]$ -GDC(6) of type 9^t with size $9t(t-1)$, where \mathcal{C}_4 is obtained by developing the elements of \mathbb{Z}_{36} in the codewords in Table 9 $+6 \pmod{36}$, \mathcal{C}_5 is obtained by developing the elements of \mathbb{Z}_{45} in the codewords in Table 9 $+3 \pmod{45}$, and \mathcal{C}_t with $t \geq 6$ is obtained by developing the elements of \mathbb{Z}_{9t} in the codewords in Table 9 $+1 \pmod{9t}$. \square

Theorem 4.4 *There is a $[3, 1]$ -GDC(6) of type 9^t with size $9t(t-1)$ for each $t \geq 4$.*

Proof: For $t \in \{4, 5, 6, 8, 9, 11, 12, 14, 15, 18, 23\}$, the desired codes are constructed in Lemma 4.3. For each $t \in \{7, 19\}$, the desired code can be obtained by inflating a $[3, 1]$ -GDC(6) of type 3^t from Lemma 4.1 with weight 3. For $t = 10$, inflate a $[3, 1]$ -GDC(6) of type 1^{10} with weight 9 to obtain the desired code. For each $t \geq 10$ and $t \notin \{10, 11, 12, 14, 15, 18, 19, 23\}$, take a $(t, \{4, 5, 6, 7, 8, 9\}, 1)$ -PBD from Theorem 2.5, and apply the Fundamental Construction with weight 9 to obtain a $[3, 1]$ -GDC(6) of type 9^t . \square

Lemma 4.5 *There is a $[3, 1]$ -GDC(6) of type 15^4 with size 300.*

Proof: Let $X = \mathbb{Z}_{60}$, and $\mathcal{G} = \{\{i, i+4, i+8, \dots, i+56\} : 0 \leq i \leq 3\}$. Then $(X, \mathcal{G}, \mathcal{C})$ is a $[3, 1]$ -GDC(6) of type 15^4 , where \mathcal{C} is obtained by developing the elements of \mathbb{Z}_{60} in the following codewords $+2 \pmod{60}$.

$$\begin{aligned} &\langle 1, 15, 40, 42 \rangle \quad \langle 0, 39, 13, 58 \rangle \quad \langle 0, 3, 34, 25 \rangle \quad \langle 0, 54, 37, 27 \rangle \quad \langle 1, 0, 7, 10 \rangle \quad \langle 1, 12, 50, 31 \rangle \\ &\langle 0, 14, 45, 23 \rangle \quad \langle 1, 56, 14, 11 \rangle \quad \langle 1, 2, 19, 52 \rangle \quad \langle 0, 55, 53, 30 \rangle \end{aligned}$$

Table 9: Base Codewords of Small $[3, 1]$ -GDC(6)s of Type 9^t in Lemma 4.3

t	Base Codewords					
4	$\langle 4, 2, 35, 29 \rangle$	$\langle 5, 19, 34, 16 \rangle$	$\langle 3, 26, 0, 21 \rangle$	$\langle 0, 15, 5, 18 \rangle$	$\langle 0, 25, 35, 34 \rangle$	$\langle 4, 15, 17, 6 \rangle$
	$\langle 5, 2, 24, 11 \rangle$	$\langle 3, 22, 16, 25 \rangle$	$\langle 2, 23, 25, 32 \rangle$	$\langle 5, 27, 26, 32 \rangle$	$\langle 4, 23, 30, 5 \rangle$	$\langle 1, 24, 18, 15 \rangle$
	$\langle 2, 13, 16, 31 \rangle$	$\langle 4, 3, 9, 18 \rangle$	$\langle 2, 28, 21, 19 \rangle$	$\langle 2, 7, 0, 9 \rangle$	$\langle 0, 1, 31, 22 \rangle$	$\langle 1, 2, 27, 20 \rangle$
5	$\langle 1, 17, 35, 9 \rangle$	$\langle 0, 16, 29, 37 \rangle$	$\langle 2, 8, 44, 1 \rangle$	$\langle 1, 34, 37, 38 \rangle$	$\langle 0, 23, 36, 19 \rangle$	$\langle 1, 32, 39, 8 \rangle$
	$\langle 0, 4, 31, 3 \rangle$	$\langle 0, 11, 44, 42 \rangle$	$\langle 1, 7, 29, 33 \rangle$	$\langle 1, 3, 20, 44 \rangle$	$\langle 0, 6, 27, 8 \rangle$	$\langle 0, 12, 34, 13 \rangle$
6	$\langle 0, 1, 35, 38 \rangle$	$\langle 0, 14, 29, 46 \rangle$	$\langle 0, 11, 52, 8 \rangle$	$\langle 0, 26, 47, 16 \rangle$	$\langle 0, 4, 9, 31 \rangle$	
8	$\langle 0, 58, 71, 19 \rangle$	$\langle 0, 55, 66, 36 \rangle$	$\langle 0, 26, 28, 67 \rangle$	$\langle 0, 43, 65, 31 \rangle$	$\langle 0, 37, 47, 52 \rangle$	$\langle 0, 3, 21, 12 \rangle$
	$\langle 0, 4, 49, 34 \rangle$					
9	$\langle 0, 14, 55, 61 \rangle$	$\langle 0, 22, 71, 20 \rangle$	$\langle 0, 8, 58, 60 \rangle$	$\langle 0, 11, 48, 77 \rangle$	$\langle 0, 53, 56, 68 \rangle$	$\langle 0, 35, 74, 69 \rangle$
	$\langle 0, 64, 80, 4 \rangle$	$\langle 0, 19, 43, 13 \rangle$				
11	$\langle 0, 40, 56, 93 \rangle$	$\langle 0, 41, 51, 98 \rangle$	$\langle 0, 25, 90, 8 \rangle$	$\langle 0, 2, 29, 71 \rangle$	$\langle 0, 45, 84, 91 \rangle$	$\langle 0, 63, 95, 26 \rangle$
	$\langle 0, 13, 31, 92 \rangle$	$\langle 0, 23, 35, 73 \rangle$	$\langle 0, 14, 19, 20 \rangle$	$\langle 0, 3, 24, 52 \rangle$		
12	$\langle 0, 80, 83, 46 \rangle$	$\langle 0, 73, 93, 107 \rangle$	$\langle 0, 44, 89, 22 \rangle$	$\langle 0, 91, 95, 52 \rangle$	$\langle 0, 10, 100, 59 \rangle$	$\langle 0, 9, 79, 85 \rangle$
	$\langle 0, 75, 77, 23 \rangle$	$\langle 0, 82, 103, 32 \rangle$	$\langle 0, 42, 53, 92 \rangle$	$\langle 0, 27, 57, 43 \rangle$	$\langle 0, 40, 101, 102 \rangle$	
14	$\langle 0, 8, 47, 77 \rangle$	$\langle 0, 45, 46, 23 \rangle$	$\langle 0, 10, 29, 63 \rangle$	$\langle 0, 91, 111, 73 \rangle$	$\langle 0, 89, 110, 41 \rangle$	$\langle 0, 3, 62, 71 \rangle$
	$\langle 0, 52, 83, 22 \rangle$	$\langle 0, 27, 93, 76 \rangle$	$\langle 0, 75, 101, 92 \rangle$	$\langle 0, 119, 124, 48 \rangle$	$\langle 0, 86, 90, 18 \rangle$	$\langle 0, 12, 94, 6 \rangle$
	$\langle 0, 13, 24, 85 \rangle$					
15	$\langle 0, 55, 116, 88 \rangle$	$\langle 0, 66, 76, 104 \rangle$	$\langle 0, 23, 123, 44 \rangle$	$\langle 0, 130, 134, 82 \rangle$	$\langle 0, 77, 85, 51 \rangle$	
	$\langle 0, 16, 78, 84 \rangle$	$\langle 0, 108, 111, 67 \rangle$	$\langle 0, 22, 42, 114 \rangle$	$\langle 0, 7, 36, 70 \rangle$	$\langle 0, 49, 89, 102 \rangle$	
	$\langle 0, 124, 133, 41 \rangle$	$\langle 0, 14, 32, 79 \rangle$	$\langle 0, 25, 96, 122 \rangle$	$\langle 0, 37, 118, 31 \rangle$		
18	$\langle 0, 28, 91, 152 \rangle$	$\langle 0, 45, 49, 52 \rangle$	$\langle 0, 70, 79, 123 \rangle$	$\langle 0, 80, 140, 115 \rangle$	$\langle 0, 46, 65, 39 \rangle$	
	$\langle 0, 37, 112, 159 \rangle$	$\langle 0, 156, 157, 20 \rangle$	$\langle 0, 73, 132, 38 \rangle$	$\langle 0, 14, 149, 81 \rangle$	$\langle 0, 42, 111, 142 \rangle$	
	$\langle 0, 104, 138, 40 \rangle$	$\langle 0, 11, 66, 95 \rangle$	$\langle 0, 32, 48, 110 \rangle$	$\langle 0, 119, 160, 129 \rangle$	$\langle 0, 56, 141, 12 \rangle$	
	$\langle 0, 139, 147, 86 \rangle$	$\langle 0, 17, 74, 150 \rangle$				
23	$\langle 0, 72, 189, 190 \rangle$	$\langle 0, 14, 116, 17 \rangle$	$\langle 0, 76, 144, 175 \rangle$	$\langle 0, 19, 169, 176 \rangle$	$\langle 0, 24, 59, 121 \rangle$	
	$\langle 0, 103, 147, 168 \rangle$	$\langle 0, 48, 101, 98 \rangle$	$\langle 0, 28, 141, 70 \rangle$	$\langle 0, 85, 96, 185 \rangle$	$\langle 0, 120, 153, 32 \rangle$	
	$\langle 0, 64, 198, 206 \rangle$	$\langle 0, 52, 177, 58 \rangle$	$\langle 0, 15, 56, 201 \rangle$	$\langle 0, 43, 123, 83 \rangle$	$\langle 0, 4, 55, 162 \rangle$	
	$\langle 0, 61, 191, 110 \rangle$	$\langle 0, 5, 133, 170 \rangle$	$\langle 0, 34, 129, 109 \rangle$	$\langle 0, 10, 36, 81 \rangle$	$\langle 0, 178, 180, 200 \rangle$	
	$\langle 0, 47, 114, 39 \rangle$	$\langle 0, 182, 194, 124 \rangle$				

□

Lemma 4.6 *There is a $[3, 1]$ -GDC(6) of type 18^4 with size 432.*

Proof: Let $X = \mathbb{Z}_{72}$, and $\mathcal{G} = \{\{i, i+4, i+8, \dots, i+68\} : 0 \leq i \leq 3\}$. Then $(X, \mathcal{G}, \mathcal{C})$ is a $[3, 1]$ -GDC(6) of type 18^4 , where \mathcal{C} is obtained by developing the elements of \mathbb{Z}_{72} in the following codewords $+1 \pmod{72}$.

$$\langle 43, 69, 22, 20 \rangle \quad \langle 62, 0, 53, 59 \rangle \quad \langle 18, 1, 55, 68 \rangle \quad \langle 31, 65, 4, 70 \rangle \quad \langle 66, 24, 25, 27 \rangle \quad \langle 0, 14, 57, 7 \rangle$$

□

Lemma 4.7 *There is a $[3, 1]$ -GDC(6) of type 39^4 with size 2028.*

Proof: Let $X = \mathbb{Z}_{156}$, and $\mathcal{G} = \{\{i, i+4, i+8, \dots, i+152\} : 0 \leq i \leq 3\}$. Then $(X, \mathcal{G}, \mathcal{C})$ is a $[3, 1]$ -GDC(6) of type 39^4 , where \mathcal{C} is obtained by developing the elements of \mathbb{Z}_{156} in the following codewords $+1 \pmod{156}$.

$$\begin{aligned} &\langle 95, 37, 0, 30 \rangle \quad \langle 112, 97, 151, 22 \rangle \quad \langle 71, 33, 122, 44 \rangle \quad \langle 8, 131, 81, 82 \rangle \quad \langle 56, 13, 98, 103 \rangle \\ &\langle 44, 57, 66, 43 \rangle \quad \langle 58, 99, 64, 113 \rangle \quad \langle 128, 49, 18, 47 \rangle \quad \langle 46, 28, 25, 35 \rangle \quad \langle 17, 76, 110, 19 \rangle \\ &\langle 23, 4, 49, 150 \rangle \quad \langle 79, 17, 148, 74 \rangle \quad \langle 0, 17, 103, 126 \rangle \end{aligned}$$

□

Lemma 4.8 *There exists a $[3, 1]$ -GDC(6) of type $9^{10}m^1$ with size $10(81 + 2m)$ for each $m \in \{18, 27\}$.*

Proof: For the type $9^{10}18^1$, let $X = I_{108}$, and $\mathcal{G} = \{\{i, i+10, i+20, \dots, i+80\} : 0 \leq i < 10\} \cup \{\{90, 91, 92, \dots, 107\}\}$. Then $(X, \mathcal{G}, \mathcal{C})$ is a $[3, 1]$ -GDC(6) of type $9^{10}18^1$, where \mathcal{C} is obtained by developing the following codewords under the automorphism group $G = \langle (0 \ 1 \ 2 \ \dots \ 89)(90 \ 91 \ \dots \ 98)(99 \ 100 \ \dots \ 107) \rangle$.

$$\begin{aligned} &\langle 90, 31, 33, 0 \rangle \quad \langle 90, 48, 26, 37 \rangle \quad \langle 90, 43, 86, 74 \rangle \quad \langle 9, 47, 82, 90 \rangle \quad \langle 99, 10, 16, 74 \rangle \quad \langle 99, 54, 8, 3 \rangle \\ &\langle 99, 6, 40, 68 \rangle \quad \langle 39, 23, 20, 99 \rangle \quad \langle 86, 14, 21, 47 \rangle \quad \langle 13, 12, 54, 66 \rangle \quad \langle 58, 37, 45, 82 \rangle \quad \langle 0, 4, 27, 36 \rangle \\ &\langle 16, 2, 31, 7 \rangle \end{aligned}$$

For the type $9^{10}27^1$, let $X = I_{117}$, and $\mathcal{G} = \{\{i, i+10, i+20, \dots, i+80\} : 0 \leq i < 10\} \cup \{\{90, 91, 92, \dots, 116\}\}$. Then $(X, \mathcal{G}, \mathcal{C})$ is a $[3, 1]$ -GDC(6) of type $9^{10}27^1$, where \mathcal{C} is obtained by developing the following codewords under the automorphism group $G = \langle (0 \ 1 \ 2 \ \dots \ 89)(90 \ 91 \ \dots \ 98)(99 \ 100 \ \dots \ 107)(108 \ 109 \ \dots \ 116) \rangle$.

$$\begin{aligned} &\langle 90, 5, 54, 26 \rangle \quad \langle 90, 13, 12, 20 \rangle \quad \langle 90, 33, 46, 25 \rangle \quad \langle 99, 11, 62, 66 \rangle \quad \langle 2, 26, 7, 90 \rangle \\ &\langle 1, 39, 37, 65 \rangle \quad \langle 99, 87, 58, 54 \rangle \quad \langle 21, 36, 79, 99 \rangle \quad \langle 108, 66, 83, 9 \rangle \quad \langle 108, 34, 78, 23 \rangle \\ &\langle 0, 23, 65, 68 \rangle \quad \langle 89, 5, 36, 108 \rangle \quad \langle 14, 77, 86, 70 \rangle \quad \langle 99, 32, 46, 43 \rangle \quad \langle 108, 28, 40, 62 \rangle \end{aligned}$$

□

Lemma 4.9 *There exists a $[3, 1]$ -GDC(6) of type $27^t 9^1$ with size $27t(3t - 1)$ for each $t \in \{4, 5, 6\}$.*

Proof: For each t , let $X_t = I_{27t+9}$, and $\mathcal{G}_t = \{\{i, i + t, i + 2t, \dots, i + 26t\} : 0 \leq i < t\} \cup \{\{27t, 27t + 1, \dots, 27t + 8\}\}$. Then $(X_t, \mathcal{G}_t, \mathcal{C}_t)$ is a $[3, 1]$ -GDC(6) of type $27^t 9^1$, where \mathcal{C}_t is obtained by developing the following codewords under the automorphism group $G = \langle (0 \ 1 \ 2 \ \dots \ 27t - 1)(27t \ 27t + 1 \ \dots \ 27t + 8) \rangle$.

$t = 4$:

$$\begin{aligned} &\langle 108, 6, 35, 57 \rangle \quad \langle 108, 16, 90, 77 \rangle \quad \langle 83, 81, 0, 6 \rangle \quad \langle 12, 67, 5, 108 \rangle \quad \langle 27, 64, 105, 10 \rangle \\ &\langle 29, 78, 52, 99 \rangle \quad \langle 108, 65, 64, 22 \rangle \quad \langle 0, 5, 94, 43 \rangle \quad \langle 12, 102, 9, 3 \rangle \quad \langle 70, 20, 59, 37 \rangle \\ &\langle 92, 19, 82, 61 \rangle \end{aligned}$$

$t = 5$:

$$\begin{aligned} &\langle 43, 0, 79, 42 \rangle \quad \langle 0, 8, 91, 84 \rangle \quad \langle 135, 49, 81, 98 \rangle \quad \langle 135, 92, 30, 14 \rangle \quad \langle 126, 58, 72, 109 \rangle \\ &\langle 135, 52, 73, 24 \rangle \quad \langle 26, 48, 72, 79 \rangle \quad \langle 80, 77, 118, 11 \rangle \quad \langle 29, 40, 31, 133 \rangle \quad \langle 33, 59, 107, 135 \rangle \\ &\langle 3, 21, 109, 37 \rangle \quad \langle 22, 35, 93, 94 \rangle \quad \langle 97, 124, 1, 130 \rangle \quad \langle 25, 48, 44, 126 \rangle \end{aligned}$$

$t = 6$:

$$\begin{aligned} &\langle 162, 26, 96, 119 \rangle \quad \langle 16, 23, 123, 104 \rangle \quad \langle 162, 59, 34, 144 \rangle \quad \langle 0, 26, 89, 13 \rangle \quad \langle 59, 39, 80, 24 \rangle \\ &\langle 75, 146, 119, 16 \rangle \quad \langle 78, 75, 118, 157 \rangle \quad \langle 162, 111, 139, 82 \rangle \quad \langle 147, 60, 2, 37 \rangle \quad \langle 53, 54, 7, 130 \rangle \\ &\langle 33, 161, 157, 52 \rangle \quad \langle 94, 108, 103, 15 \rangle \quad \langle 89, 134, 156, 145 \rangle \quad \langle 37, 70, 135, 86 \rangle \quad \langle 86, 18, 49, 33 \rangle \\ &\langle 0, 50, 101, 130 \rangle \quad \langle 83, 73, 81, 162 \rangle \end{aligned}$$

□

Lemma 4.10 *There exists a $[3, 1]$ -GDC(6) of type $27^t 18^1$ with size $27t(3t + 1)$ for each $t \in \{4, 6\}$.*

Proof: For each t , let $X_t = I_{27t+18}$, and $\mathcal{G}_t = \{\{i, i + t, i + 2t, \dots, i + 26t\} : 0 \leq i < t\} \cup \{\{27t, 27t + 1, \dots, 27t + 17\}\}$. Then $(X_t, \mathcal{G}_t, \mathcal{C}_t)$ is a $[3, 1]$ -GDC(6) of type $27^t 18^1$, where \mathcal{C}_t is obtained by developing the following codewords under the automorphism group $G = \langle (0 \ 1 \ 2 \ \dots \ 27t - 1)(27t \ 27t + 1 \ \dots \ 27t + 8)(27t + 9 \ 27t + 10 \ \dots \ 27t + 17) \rangle$.

$t = 4$:

$$\begin{aligned} &\langle 108, 31, 86, 96 \rangle \quad \langle 108, 46, 21, 7 \rangle \quad \langle 108, 26, 9, 20 \rangle \quad \langle 97, 47, 96, 108 \rangle \quad \langle 37, 15, 28, 18 \rangle \\ &\langle 117, 88, 55, 94 \rangle \quad \langle 117, 35, 5, 20 \rangle \quad \langle 54, 9, 27, 16 \rangle \quad \langle 117, 12, 54, 33 \rangle \quad \langle 84, 43, 38, 81 \rangle \\ &\langle 31, 29, 60, 117 \rangle \quad \langle 53, 6, 88, 107 \rangle \quad \langle 0, 23, 74, 37 \rangle \end{aligned}$$

$t = 6$:

$$\begin{aligned} &\langle 162, 22, 110, 0 \rangle \quad \langle 162, 24, 158, 46 \rangle \quad \langle 171, 110, 161, 69 \rangle \quad \langle 18, 35, 82, 162 \rangle \\ &\langle 171, 139, 30, 46 \rangle \quad \langle 136, 135, 13, 120 \rangle \quad \langle 171, 160, 149, 81 \rangle \quad \langle 99, 42, 91, 171 \rangle \\ &\langle 162, 79, 3, 134 \rangle \quad \langle 39, 119, 114, 128 \rangle \quad \langle 128, 95, 154, 141 \rangle \quad \langle 118, 12, 83, 74 \rangle \\ &\langle 143, 106, 39, 54 \rangle \quad \langle 156, 136, 113, 86 \rangle \quad \langle 131, 135, 160, 66 \rangle \quad \langle 48, 50, 29, 129 \rangle \\ &\langle 91, 84, 81, 125 \rangle \quad \langle 60, 105, 137, 91 \rangle \quad \langle 0, 38, 99, 65 \rangle \end{aligned}$$

□

Lemma 4.11 *There exists a $[3, 1]$ -GDC(6) of type $36^6 27^1$ with size 5616.*

Proof: Let $X = I_{243}$, and $\mathcal{G} = \{\{i, i+6, i+12, \dots, i+210\} : 0 \leq i < 6\} \cup \{\{216, 217, \dots, 242\}\}$. Then $(X, \mathcal{G}, \mathcal{C})$ is a $[3, 1]$ -GDC(6) of type $36^6 27^1$, where \mathcal{C} is obtained by developing the following codewords under the automorphism group $G = \langle (0 \ 1 \ 2 \ \dots \ 215)(216 \ 217 \ \dots \ 224)(225 \ 226 \ \dots \ 233)(234 \ 235 \ \dots \ 242) \rangle$.

$$\begin{array}{llll} \langle 33, 34, 92, 41 \rangle & \langle 14, 148, 61, 107 \rangle & \langle 159, 204, 154, 61 \rangle & \langle 184, 211, 162, 165 \rangle \\ \langle 199, 99, 0, 89 \rangle & \langle 175, 80, 11, 190 \rangle & \langle 134, 42, 173, 216 \rangle & \langle 225, 183, 107, 145 \rangle \\ \langle 0, 25, 105, 119 \rangle & \langle 159, 96, 73, 116 \rangle & \langle 225, 160, 23, 186 \rangle & \langle 175, 105, 208, 225 \rangle \\ \langle 152, 27, 214, 7 \rangle & \langle 34, 135, 146, 131 \rangle & \langle 225, 153, 92, 157 \rangle & \langle 234, 118, 149, 159 \rangle \\ \langle 216, 34, 91, 26 \rangle & \langle 170, 103, 29, 156 \rangle & \langle 234, 124, 207, 89 \rangle & \langle 166, 134, 132, 234 \rangle \\ \langle 216, 14, 54, 51 \rangle & \langle 142, 198, 121, 33 \rangle & \langle 234, 200, 48, 193 \rangle & \langle 181, 209, 165, 200 \rangle \\ \langle 7, 20, 155, 129 \rangle & \langle 216, 129, 184, 56 \rangle & & \end{array}$$

□

Lemma 4.12 *There is a $[3, 1]$ -GDC(6) of type $36^u(9x)^1$ with size $72u(2u+x-2)$ for $u \geq 4$, $u \notin \{6, 10\}$ and $0 \leq x \leq u$.*

Proof: Take a $\text{TD}(5, u)$ with $u \geq 4$, $u \notin \{6, 10\}$ from Theorem 2.7; remove one point from one group and adjoin an ideal point to obtain a $\{5, u+1\}$ -GDD of type $4^u u^1$. Apply the Fundamental Construction with weight 9 to the points in the groups of size 4 and x points in the group of size u for $0 \leq x \leq u$. Note that there exist $[3, 1]$ -GDC(6)s of type 9^s for $s \geq 4$ by Theorem 4.4. The result is the desired GDC. □

Lemma 4.13 *The following $[3, 1]$ -GDC(6)s all exist:*

- i) type $45^5(9x)^1$ and size $450(10+x)$ for $x \in \{0, 1, 2\}$;
- ii) type $36^6(9x)^1$ and size $432(10+x)$ for $x \in \{0, 1, 3\}$;
- iii) type $36^{10}(9x)^1$ and size $720(18+x)$ for $x \in \{0, 1, 2, 3\}$;
- iv) type 27^t and size $81t(t-1)$ for $t \in \{4, 5, 7\}$;
- v) type 18^7 and size 1512.

Proof: For i), take a $\text{TD}(6, 5)$ from Theorem 2.7, and apply the Fundamental Construction with weight 9 to all the points in the first five groups and x points in the last group. For ii), take a $[3, 1]$ -GDC(6) of type 9^6 from Theorem 4.4; apply the Inflation Construction with weight 4, using 4-MGDDs of type 4^4 as input designs, to obtain a 4-DGDD of type $(36, 9^4)^6$ with the CCC property. Then adjoin $9x$ ideal points with $x \in \{0, 1\}$ and fill in

$[3, 1]$ -GDC(6)s of type $9^6(9x)^1$ to obtain a $[3, 1]$ -GDC(6) of type $36^6(9x)^1$, as desired; for $x = 3$, see Lemma 4.11. For iii), we proceed similarly, starting instead with a $[3, 1]$ -GDC(6) of type 9^{10} to obtain a 4-DGDD of type $(36, 9^4)^{10}$ with the CCC property. Then adjoin $9x$ ideal points and fill in $[3, 1]$ -GDC(6)s of type $9^{10}(9x)^1$ from Theorem 4.4 and Lemma 4.8 to obtain the desired GDCs. For iv), inflate a $[3, 1]$ -GDC(6) of type 9^t from Theorem 4.4 with weight 3. For v), apply the Fundamental Construction with weight 9 to a 4-GDD of type 2^7 (see [26]) to obtain the desired GDC. \square

4.2 Cases of Length $n \equiv 0, 1, 2, 3 \pmod{9}$

Lemma 4.14 $A_3(9t + 1, 6, [3, 1]) = U(9t + 1, 6, [3, 1])$ for $t \in \{1, 3\}$.

Proof: For $t = 1$, see Table 1. For $t = 3$, let $X = \mathbb{Z}_{28}$. Then (X, \mathcal{C}) is the desired optimal $(28, 6, [3, 1])_3$ -code, where \mathcal{C} is obtained by developing the following base codewords $+4 \pmod{28}$.

$$\begin{array}{cccccc} \langle 1, 3, 26, 8 \rangle & \langle 2, 8, 18, 26 \rangle & \langle 2, 16, 17, 22 \rangle & \langle 3, 16, 18, 7 \rangle & \langle 0, 8, 25, 21 \rangle & \langle 1, 21, 22, 16 \rangle \\ \langle 2, 3, 23, 6 \rangle & \langle 0, 3, 24, 12 \rangle & \langle 1, 17, 27, 23 \rangle & \langle 0, 9, 26, 23 \rangle & \langle 3, 19, 20, 25 \rangle & \langle 3, 13, 22, 17 \rangle \end{array}$$

\square

Lemma 4.15 $A_3(9t, 6, [3, 1]) = U(9t, 6, [3, 1])$ for each $t \in \{1, 2, 3\}$.

Proof: For each $t \in \{1, 3\}$, the desired code can be shorten from an optimal $(9t+1, 6, [3, 1])_3$ -code. For $t = 2$, let $X = \mathbb{Z}_{18}$. Then an $(18, 6, [3, 1])_3$ -code with 30 codewords is obtained by developing the codewords $\langle 0, 1, 2, 3 \rangle$, $\langle 8, 11, 13, 2 \rangle$, $\langle 0, 7, 10, 16 \rangle$, $\langle 0, 5, 12, 9 \rangle$, and $\langle 2, 12, 16, 8 \rangle$ $+3 \pmod{18}$. \square

Lemma 4.16 *There exists a $[3, 1]$ -GDC(6) of type $1^9 2^1$ with size 11 and a $[3, 1]$ -GDC(6) of type $1^9 3^1$ with size 12.*

Proof: The codewords for each desired code constructed on I_{11} or I_{12} are listed below.

$1^9 2^1$:

$$\begin{array}{cccccc} \langle 1, 3, 9, 0 \rangle & \langle 1, 4, 7, 8 \rangle & \langle 2, 8, 9, 4 \rangle & \langle 5, 8, 10, 1 \rangle & \langle 0, 4, 5, 9 \rangle & \langle 1, 2, 6, 10 \rangle \\ \langle 0, 7, 10, 2 \rangle & \langle 2, 3, 5, 7 \rangle & \langle 0, 6, 8, 3 \rangle & \langle 3, 4, 10, 6 \rangle & \langle 6, 7, 9, 5 \rangle & \end{array}$$

$1^9 3^1$:

$$\begin{array}{cccccc} \langle 3, 7, 11, 0 \rangle & \langle 2, 4, 7, 10 \rangle & \langle 7, 8, 9, 1 \rangle & \langle 0, 2, 9, 3 \rangle & \langle 0, 1, 10, 7 \rangle & \langle 6, 8, 10, 4 \rangle \\ \langle 0, 5, 8, 11 \rangle & \langle 1, 4, 11, 8 \rangle & \langle 1, 3, 6, 9 \rangle & \langle 4, 5, 9, 6 \rangle & \langle 2, 6, 11, 5 \rangle & \langle 3, 5, 10, 2 \rangle \end{array}$$

\square

Lemma 4.17 $A_3(9t + 2, 6, [3, 1]) = U(9t + 2, 6, [3, 1])$ for $t \in \{1, 2, 3\}$.

Proof: For $t = 1$, the $[3, 1]$ -GDC of type $1^9 2^1$ constructed in Lemma 4.16 is the desired code. For each $t \in \{2, 3\}$, let $X_t = \mathbb{Z}_{9t+2}$. Then (X_t, \mathcal{C}_t) is the desired optimal $(9t + 2, 6, [3, 1])_3$ -code, where \mathcal{C}_t is obtained by developing the elements of \mathbb{Z}_{9t+2} in the following codewords $+1 \pmod{9t+2}$.

$$t = 2: \langle 0, 1, 9, 15 \rangle \langle 0, 3, 7, 5 \rangle$$

$$t = 3: \langle 0, 1, 26, 15 \rangle \langle 0, 6, 13, 11 \rangle \langle 0, 8, 20, 10 \rangle$$

□

Lemma 4.18 $A_3(9t + 3, 6, [3, 1]) = U(9t + 3, 6, [3, 1])$ for each $t \in \{1, 2, 3\}$.

Proof: For $t = 1$, the $[3, 1]$ -GDC of type $1^9 3^1$ constructed in Lemma 4.16 is the desired code. For $t = 2$, the $[3, 1]$ -GDC(6) of type 3^7 constructed in Lemma 4.1 is the desired code. For $t = 3$, inflate a $[3, 1]$ -GDC(6) of type 1^{10} with weight 3 to obtain a $[3, 1]$ -GDC(6) of type 3^{10} , which is the desired code. □

Theorem 4.19 $A_3(9t + i, 6, [3, 1]) = U(9t + i, 6, [3, 1])$ for each $t \geq 1$ and $i \in \{0, 1, 2, 3\}$, except possibly for $(t, i) = (2, 1)$.

Proof: For $t \leq 3$, see Lemmas 4.14–4.15, 4.17 and 4.18. For each $t \geq 4$, take a $[3, 1]$ -GDC(6) of type 9^t constructed in Theorem 4.4. Adjoin i ideal points and fill in the groups with $[3, 1]$ -GDC(6)s of type $1^9 i^1$ constructed in Lemmas 4.14, 4.15 and 4.16 to obtain the desired optimal $(9t + i, 6, [3, 1])_3$ -code. □

4.3 Case of Length $n \equiv 6 \pmod{9}$

Lemma 4.20 There is a $[3, 1]$ -GDC(6) of type $1^{9t} 6^1$ with size $9t(t+1)$ for each $t \in \{2, 3, 5, 7, 9\}$.

Proof: Let $X_t = I_{9t+6}$, and $\mathcal{G}_t = \{\{i\} : 0 \leq i \leq 9t - 1\} \cup \{\{9t, 9t + 1, \dots, 9t + 5\}\}$. Then $(X_t, \mathcal{G}_t, \mathcal{C}_t)$ is a $[3, 1]$ -GDC(6) of type $1^{9t} 6^1$, where \mathcal{C}_t is obtained by developing the codewords in Table 10 under the automorphism group G_t .

$$\text{For } t \in \{2, 3\}, G_t = \langle (0 \ 3 \ 6 \ \dots \ 9t-3)(1 \ 4 \ 7 \ \dots \ 9t-2)(2 \ 5 \ 8 \ \dots \ 9t-1)(9t \ 9t+1 \ 9t+2)(9t+3 \ 9t+4 \ 9t+5) \rangle.$$

$$\text{For } t \in \{5, 7, 9\}, G_t = \langle (0 \ 1 \ 2 \ \dots \ 9t-1)(9t \ 9t+1 \ 9t+2)(9t+3 \ 9t+4 \ 9t+5) \rangle.$$

□

Lemma 4.21 $A_3(9t + 6, 6, [3, 1]) = U(9t + 6, 6, [3, 1])$ for $0 \leq t \leq 10$.

Table 10: Base Codewords of Small $[3, 1]$ -GDC(6)s of type $1^{9t}6^1$ in Lemma 4.20

t	Codewords
2	$\langle 0, 1, 19, 2 \rangle \langle 1, 3, 22, 16 \rangle \langle 0, 7, 23, 14 \rangle \langle 2, 21, 5, 6 \rangle \langle 1, 9, 20, 6 \rangle \langle 2, 18, 14, 1 \rangle \langle 0, 8, 4, 17 \rangle$ $\langle 0, 6, 11, 9 \rangle \langle 2, 10, 4, 13 \rangle$
3	$\langle 1, 6, 17, 30 \rangle \langle 1, 31, 14, 9 \rangle \langle 0, 20, 32, 16 \rangle \langle 0, 29, 25, 5 \rangle \langle 1, 29, 22, 15 \rangle \langle 2, 11, 1, 12 \rangle$ $\langle 2, 23, 4, 26 \rangle \langle 0, 21, 18, 17 \rangle \langle 2, 17, 29, 21 \rangle \langle 2, 15, 0, 7 \rangle \langle 0, 4, 30, 8 \rangle \langle 0, 1, 10, 13 \rangle$
5	$\langle 0, 39, 15, 20 \rangle \langle 0, 35, 36, 33 \rangle \langle 0, 13, 41, 8 \rangle \langle 0, 31, 49, 38 \rangle \langle 0, 47, 23, 25 \rangle \langle 0, 11, 29, 3 \rangle$
7	$\langle 0, 5, 21, 53 \rangle \langle 0, 20, 65, 7 \rangle \langle 0, 25, 54, 31 \rangle \langle 0, 2, 3, 15 \rangle \langle 0, 55, 68, 44 \rangle \langle 0, 33, 37, 56 \rangle$ $\langle 0, 27, 41, 51 \rangle \langle 0, 17, 35, 11 \rangle$
9	$\langle 0, 19, 58, 28 \rangle \langle 0, 6, 32, 44 \rangle \langle 0, 33, 50, 30 \rangle \langle 0, 41, 85, 52 \rangle \langle 0, 4, 67, 20 \rangle \langle 0, 8, 82, 7 \rangle$ $\langle 0, 2, 15, 3 \rangle \langle 0, 25, 35, 72 \rangle \langle 0, 5, 27, 70 \rangle \langle 0, 21, 45, 74 \rangle$

Proof: For $t = 0$, see Table 1. For $t = 1$, let $X = \mathbb{Z}_{15}$. Then (X, \mathcal{C}) is the desired optimal $(15, 6, [3, 1])_3$ -code, where \mathcal{C} is obtained by developing the codewords $\langle 2, 5, 0, 7 \rangle$, $\langle 1, 8, 2, 9 \rangle$, $\langle 0, 4, 9, 8 \rangle$, and $\langle 1, 13, 0, 11 \rangle + 3 \pmod{15}$. For each $t \in \{4, 6, 8, 10\}$, take a $[3, 1]$ -GDC(6) of type $6^{3t/2+1}$ from Lemma 4.2, and fill in the groups with optimal $(6, 6, [3, 1])_3$ -codes to obtain the desired code. For each $t \in \{2, 3, 5, 7, 9\}$, take a $[3, 1]$ -GDC(6) of type $1^{9t}6^1$ from Lemma 4.20, and fill in the group of size 6 with an optimal $(6, 6, [3, 1])_3$ -code to obtain the desired code. \square

Lemma 4.22 $A_3(9t + 6, 6, [3, 1]) = U(9t + 6, 6, [3, 1])$ for $t \geq 11$.

Proof: For each $t \geq 16$ and $t \neq 26$, write $t = 4u + x$ with $u \geq 4$ and $x \in \{0, 1, 2, 3\}$. Take a $[3, 1]$ -GDC(6) of type $36^u(9x)^1$ from Lemmas 4.12–4.13. Adjoin six ideal points, and fill in the groups of size 36 together with the ideal points with $[3, 1]$ -GDC(6)s of type 6^7 from Lemma 4.2 to obtain a $[3, 1]$ -GDC(6) of type $6^{6u}(9x + 6)^1$. Then fill in the groups with optimal $(6, 6, [3, 1])_3$ -codes and an optimal $(9x + 6, 6, [3, 1])_3$ -code from Lemma 4.21 to obtain an optimal $(36u + 9x + 6, 6, [3, 1])_3$ -code, as desired. For $t = 26$, take a $[3, 1]$ -GDC(6) of type $45^5 9^1$ from Lemma 4.13. Adjoin 6 ideal points, fill in the groups of size 45 together with these ideal points with $[3, 1]$ -GDC(6)s of type $1^{45}6^1$ from Lemma 4.20, and fill in the group of size 9 together with these ideal points with an optimal $(15, 6, [3, 1])_3$ -code to obtain the desired code.

For each $t \in \{12, 14, 15\}$, take a $[3, 1]$ -GDC(6) of type 27^4 , type 18^7 , or type 27^5 from Lemma 4.13. Adjoin 6 ideal points and fill in the groups to obtain the desired code. For $t = 13$, we proceed similarly, starting instead with a $[3, 1]$ -GDC(6) of type $27^4 9^1$ from Lemma 4.9.

For $t = 11$, inflate a $[3, 1]$ -GDC(6) of type 3^7 from Lemma 4.1 with weight 5 to obtain a $[3, 1]$ -GDC(6) of type 15^7 . Fill in the groups with optimal $(15, 6, [3, 1])_3$ -codes to complete the proof. \square

Combining Lemmas 4.21 and 4.22, we have the following result.

Theorem 4.23 $A_3(9t + 6, 6, [3, 1]) = U(9t + 6, 6, [3, 1])$ for each $t \geq 0$.

4.4 Case of Length $n \equiv 4 \pmod{9}$

Lemma 4.24 There exists a $[3, 1]$ -GDC(6) of type 4^{9t+1} with size $16t(9t + 1)$ for each $t \in \{1, 2, 3\}$.

Proof: For $t = 2$, let $X = \mathbb{Z}_{76}$, and $\mathcal{G} = \{\{0, 19, 38, 57\} + i : 0 \leq i \leq 18\}$. Then $(X, \mathcal{G}, \mathcal{C})$ is a $[3, 1]$ -GDC(6) of type 4^{19} , where \mathcal{C} is obtained by developing the elements of \mathbb{Z}_{76} in the following codewords $+1 \pmod{76}$.

$$\begin{array}{cccc} \langle 0, 26, 22, 9 \rangle & \langle 0, 62, 74, 13 \rangle & \langle 0, 37, 6, 67 \rangle & \langle 0, 44, 65, 41 \rangle \\ \langle 0, 5, 56, 29 \rangle & \langle 0, 43, 53, 46 \rangle & \langle 0, 42, 60, 1 \rangle & \langle 0, 8, 36, 7 \rangle \end{array}$$

For each $t \in \{1, 3\}$, inflate a $[3, 1]$ -GDC(6) of type 1^{9t+1} from Lemma 4.14 with weight 4 to obtain the desired code. \square

Lemma 4.25 $A_3(9t + 4, 6, [3, 1]) = U(9t + 4, 6, [3, 1])$ for $t \in \{0, 2, 3, 4, 8, 12\}$.

Proof: For $t = 0$, see Table 1.

For $t \in \{2, 3\}$, let $X_t = \mathbb{Z}_{9t+3} \cup \{\infty\}$. Then (X_t, \mathcal{C}_t) is the desired optimal $(9t+4, 6, [3, 1])_3$ -code, where \mathcal{C}_t is developed $+3 \pmod{9t+3}$.

$$t = 2: \langle 1, 6, 12, 5 \rangle \langle 0, 12, 8, 19 \rangle \langle 1, 8, 18, 15 \rangle \langle 0, 1, 2, 3 \rangle \langle 2, 5, 11, 7 \rangle \langle 1, 10, 7, 20 \rangle \langle 0, 5, \infty, 13 \rangle$$

$$t = 3: \langle 0, 2, 16, 6 \rangle \langle 0, 1, 25, 23 \rangle \langle 0, 7, 28, 15 \rangle \langle 2, 5, 22, 4 \rangle \langle 2, 7, 18, 26 \rangle \langle 2, 21, 11, 17 \rangle \langle 0, 9, 12, 22 \rangle \\ \langle 2, 28, 25, 13 \rangle \langle 0, 17, 29, 24 \rangle \langle 0, \infty, 4, 5 \rangle$$

For $t \in \{4, 8, 12\}$, take $[3, 1]$ -GDC(6)s of type $4^{9t/4+1}$ from Lemma 4.24, and fill in the groups with optimal $(4, 6, [3, 1])_3$ -codes to obtain the desired codes. \square

Lemma 4.26 $A_3(9t + 4, 6, [3, 1]) = U(9t + 4, 6, [3, 1])$ for each $t \not\equiv 1 \pmod{4}$, $t \geq 16$ and $t \neq 26$.

Proof: For each $t \not\equiv 1 \pmod{4}$, $t \geq 16$ and $t \neq 26$, write $t = 4u + x$ such that $x \in \{0, 2, 3\}$. Take a $[3, 1]$ -GDC(6) of type $36^u(9x)^1$ from Lemmas 4.12–4.13, adjoin four ideal points, and fill in the groups of size 36 together with these ideal points with $[3, 1]$ -GDC(6)s of type 4^{10} from Lemma 4.24 to obtain a $[3, 1]$ -GDC(6) of type $4^{9u}(9x + 4)^1$. Fill in the groups with optimal $(4, 6, [3, 1])_3$ -codes and an optimal $(9x + 4, 6, [3, 1])_3$ -code from Lemma 4.25 to obtain an optimal $(36u + 9x + 4, 6, [3, 1])_3$ -code, as desired. \square

Lemma 4.27 $A_3(9t+4, 6, [3, 1]) = U(9t+4, 6, [3, 1])$ for $t \in \{17, 33\}$ or $t \equiv 1 \pmod{4}, t \geq 85$.

Proof: For $t = 17$, take a $[3, 1]$ -GDC(6) of type 39^4 from Lemma 4.7 and adjoin an ideal point. Then fill in the groups together with the ideal point with optimal codes of length 40 from Lemma 4.25 to obtain the desired code.

For $t = 33$, take a TD(4, 5) and assign weight 15 to each point of this TD. Note that there exists a $[3, 1]$ -GDC(6) of type 15^4 by Lemma 4.5. We obtain a $[3, 1]$ -GDC(6) of type 75^4 . Now adjoin an ideal point and fill in the groups together with the ideal point with optimal codes of length 76 to obtain the desired code.

For $t \geq 85$, write $t = 4u + 17$ with $u \geq 17$. Take a $[3, 1]$ -GDC(6) of type $36^u 153^1$ from Lemma 4.12, adjoin four ideal points, and fill in the groups to obtain an optimal $(36u + 157, 6, [3, 1])_3$ -code. \square

Summarizing the above results, we have:

Theorem 4.28 $A_3(9t+4, 6, [3, 1]) = U(9t+4, 6, [3, 1])$ for each $t \geq 0$ and $t \notin \{1, 5, 6, 7, 9, 10, 11, 13, 14, 15, 21, 25, 26, 29, 37, 41, 45, 49, 53, 57, 61, 65, 69, 73, 77, 81\}$.

4.5 Case of Length $n \equiv 5 \pmod{9}$

Lemma 4.29 There exists a $[3, 1]$ -GDC(6) of type $1^{36}5^1$ with 173 codewords.

Proof: Let $X = (\mathbb{Z}_{12} \times \{0, 1, 2\}) \cup (\mathbb{Z}_3 \times \{3\}) \cup \{\infty_0, \infty_1\}$. The point set $(\mathbb{Z}_3 \times \{3\}) \cup \{\infty_0, \infty_1\}$ forms the group of size 5. Define $\alpha : X \rightarrow X$ as $x_y \rightarrow (x+1)_y$ where the addition is modulo 12 if $y \in \{0, 1, 2\}$, and modulo 3 if $y = 3$. The points ∞_0 and ∞_1 are fixed by α . Develop the following 13 base codewords with α :

$$\begin{array}{lllll} \langle 0_3, 6_0, 2_2, 0_2 \rangle & \langle 3_2, 11_1, 7_2, 0_3 \rangle & \langle 10_2, 10_1, 3_2, 1_0 \rangle & \langle 4_2, 0_0, 9_0, 7_1 \rangle & \langle 8_0, 10_0, 8_2, 7_2 \rangle \\ \langle 1_0, 0_1, 2_2, 3_2 \rangle & \langle 0_3, 4_0, 11_0, 8_1 \rangle & \langle 1_1, 11_1, 10_0, 9_0 \rangle & \langle 0_1, 1_2, 7_1, 2_0 \rangle & \langle \infty_0, 2_1, 1_2, 7_0 \rangle \\ \langle 0_3, 3_1, 4_1, 1_2 \rangle & \langle \infty_1, 9_2, 4_0, 6_1 \rangle & \langle 1_1, 5_0, 8_2, 10_1 \rangle & & \end{array}$$

Further develop the codeword $\langle 0_0, 6_0, 0_1, 6_1 \rangle$ making a short orbit of length 6.

Finally develop the following codewords making 2 short orbits of length 4 and one short orbit of length 3:

$$\langle 0_0, 4_0, 8_0, \infty_0 \rangle \quad \langle 0_1, 4_1, 8_1, \infty_1 \rangle \quad \langle 0_2, 3_2, 6_2, 9_2 \rangle$$

\square

Lemma 4.30 $A_3(9t+5, 6, [3, 1]) = U(9t+5, 6, [3, 1])$ for $t \in \{0, 1, 2, 3, 4\}$.

Proof: For $t = 0$, see Table 1.

For $t = 1$, an optimal $(14, 6, [3, 1])_3$ -code is constructed on I_{14} with 17 codewords listed below.

$$\begin{array}{cccccc} \langle 0, 4, 9, 5 \rangle & \langle 7, 4, 10, 6 \rangle & \langle 2, 10, 9, 3 \rangle & \langle 13, 10, 0, 8 \rangle & \langle 2, 12, 8, 4 \rangle & \langle 0, 1, 7, 12 \rangle \\ \langle 3, 5, 8, 0 \rangle & \langle 2, 5, 7, 13 \rangle & \langle 6, 9, 8, 11 \rangle & \langle 3, 6, 12, 10 \rangle & \langle 3, 13, 4, 2 \rangle & \langle 12, 13, 9, 7 \rangle \\ \langle 2, 11, 0, 6 \rangle & \langle 11, 4, 1, 8 \rangle & \langle 11, 3, 7, 9 \rangle & \langle 11, 10, 5, 12 \rangle & \langle 13, 1, 6, 5 \rangle & \end{array}$$

For $t = 2$, the desired code is constructed on $\mathbb{Z}_{21} \cup \{\infty_1, \infty_2\}$ and is obtained by developing the elements of \mathbb{Z}_{21} in the following codewords $+7 \pmod{21}$, where the infinite points ∞_0, ∞_1 are fixed when developed.

$$\begin{array}{cccccc} \langle 0, 1, 2, 3 \rangle & \langle 9, 20, 16, 3 \rangle & \langle 7, 16, 11, 19 \rangle & \langle 5, 8, 19, 2 \rangle & \langle 2, 12, 7, 20 \rangle & \langle 0, 11, 18, 10 \rangle \\ \langle 10, 5, 4, 6 \rangle & \langle 15, 3, 10, 2 \rangle & \langle 1, 14, 20, \infty_2 \rangle & \langle 6, 7, 3, 14 \rangle & \langle 2, \infty_2, 4, 15 \rangle & \langle 9, \infty_1, 18, 15 \rangle \\ \langle 1, 7, 5, \infty_1 \rangle & \langle \infty_2, 12, 3, 0 \rangle & \langle 10, \infty_1, 20, 12 \rangle & \langle 8, 1, 13, 11 \rangle & \langle 13, 18, 6, 12 \rangle & \end{array}$$

For $t = 3$, the desired code is constructed on \mathbb{Z}_{32} and is obtained by developing the elements of \mathbb{Z}_{32} in the following codewords $+4 \pmod{32}$.

$$\begin{array}{cccccc} \langle 2, 8, 31, 1 \rangle & \langle 2, 9, 24, 23 \rangle & \langle 2, 14, 17, 30 \rangle & \langle 1, 12, 23, 2 \rangle & \langle 3, 20, 22, 26 \rangle & \langle 0, 18, 29, 13 \rangle \\ \langle 0, 28, 8, 3 \rangle & \langle 2, 26, 27, 28 \rangle & \langle 2, 20, 25, 21 \rangle & \langle 3, 1, 27, 31 \rangle & \langle 3, 23, 18, 28 \rangle & \langle 3, 17, 29, 21 \rangle \\ \langle 0, 9, 19, 16 \rangle & & & & & \end{array}$$

Finally, for $t = 4$, take the $[3, 1]$ -GDC(6) of type $1^{36}5^1$ constructed in Lemma 4.29 and fill in the group of size 5 with an optimal code of length 5. \square

Lemma 4.31 $A_3(9t + 5, 6, [3, 1]) = U(9t + 5, 6, [3, 1])$ for each $t \geq 16$ and $t \neq 26$.

Proof: For each $t \geq 16$ and $t \neq 26$, write $t = 4u + x$ such that $0 \leq x \leq 3$. Take a $[3, 1]$ -GDC(6) of type $36^u(9x)^1$ from Lemmas 4.12 and 4.13. Adjoin five ideal points, and fill in the groups of size 36 together with the ideal points with $[3, 1]$ -GDC(6)s of type $1^{36}5^1$ from Lemma 4.29 to obtain a $[3, 1]$ -GDC(6) of type $1^{36u}(9x + 5)^1$. Then fill in the group with an optimal code of length $9x + 5$ from Lemma 4.30 to obtain an optimal code of length $9(4u + x) + 5$, as desired. \square

Theorem 4.32 $A_3(9t + 5, 6, [3, 1]) = U(9t + 5, 6, [3, 1])$ for each $t \geq 0$ and $t \notin [5, 15] \cup \{26\}$.

4.6 Case of Length $n \equiv 7 \pmod{9}$

Lemma 4.33 There exists a $[3, 1]$ -GDC(6) of type $1^{36}7^1$ with 190 codewords.

Proof: Let $X = (\mathbb{Z}_{12} \times \{0, 1, 2\}) \cup (\mathbb{Z}_3 \times \{3, 4\}) \cup \{\infty\}$. The point set $(\mathbb{Z}_3 \times \{3, 4\}) \cup \{\infty\}$ forms the group of size 7. Define $\alpha : X \rightarrow X$ as $x_y \rightarrow (x+1)_y$ where the addition is modulo 12 if $y \in \{0, 1, 2\}$, and modulo 3 if $y \in \{3, 4\}$. The point ∞ is fixed by α . Develop the following 15 base codewords with α :

$$\begin{array}{ccccc} \langle 0_3, 4_2, 6_1, 8_0 \rangle & \langle 0_3, 9_2, 4_1, 8_1 \rangle & \langle 0_3, 5_2, 7_0, 0_0 \rangle & \langle 8_0, 6_0, 11_1, 0_3 \rangle & \langle 0_4, 6_1, 11_1, 6_2 \rangle \\ \langle 0_4, 0_0, 1_1, 7_2 \rangle & \langle 0_4, 8_2, 7_0, 8_0 \rangle & \langle 0_2, 8_0, 7_2, 0_4 \rangle & \langle 0_0, 3_0, 11_1, 3_2 \rangle & \langle 3_0, 0_2, 10_1, 1_1 \rangle \\ \langle 8_1, 4_0, 6_1, 9_2 \rangle & \langle 6_0, 2_0, 8_2, 1_0 \rangle & \langle \infty, 4_0, 1_1, 0_2 \rangle & \langle 8_2, 10_2, 7_2, 4_1 \rangle & \langle 10_1, 7_2, 11_1, 7_1 \rangle \end{array}$$

Further develop the following codewords making a short orbit of length 6 and a short orbit of length 4:

$$\langle 0_0, 6_0, 0_1, 6_1 \rangle \quad \langle 0_2, 4_2, 8_2, \infty \rangle$$

□

Lemma 4.34 *There exists a $[3, 1]$ -GDC(6) of type $1^{54}7^1$ with 393 codewords.*

Proof: Let $X = (\mathbb{Z}_{18} \times \{0, 1, 2\}) \cup (\mathbb{Z}_3 \times \{3, 4\}) \cup \{\infty\}$. The point set $(\mathbb{Z}_3 \times \{3, 4\}) \cup \{\infty\}$ forms the group of size 7. Define $\alpha : X \rightarrow X$ as $x_y \rightarrow (x+1)_y$ where the addition is modulo 18 if $y \in \{0, 1, 2\}$, and modulo 3 if $y \in \{3, 4\}$. The point ∞ is fixed by α . Develop the following 21 base codewords with α :

$$\begin{array}{ccccc} \langle 3_1, 5_2, 8_2, 4_0 \rangle & \langle 0_4, 1_2, 8_0, 15_2 \rangle & \langle 0_3, 17_2, 8_1, 10_0 \rangle & \langle 0_0, 2_1, 13_2, 12_1 \rangle & \langle 0_4, 3_0, 13_0, 11_2 \rangle \\ \langle 3_0, 5_0, 9_1, 7_2 \rangle & \langle 14_2, 1_2, 2_0, 5_2 \rangle & \langle 1_2, 2_2, 13_1, 17_0 \rangle & \langle 0_3, 4_1, 11_0, 12_2 \rangle & \langle 11_0, 12_0, 5_0, 10_1 \rangle \\ \langle 0_0, 5_0, 8_1, 1_1 \rangle & \langle 0_5, 7_0, 4_1, 17_2 \rangle & \langle 6_0, 3_2, 11_2, 13_1 \rangle & \langle 0_3, 13_2, 9_1, 15_0 \rangle & \langle 1_1, 13_2, 14_1, 15_2 \rangle \\ \langle 6_1, 3_1, 3_2, 11_0 \rangle & \langle 0_4, 10_1, 9_1, 17_1 \rangle & \langle 3_0, 12_2, 6_0, 16_1 \rangle & \langle 15_2, 15_0, 1_0, 0_3 \rangle & \langle 0_1, 12_1, 16_1, 11_0 \rangle \\ \langle 8_1, 0_2, 11_2, 0_4 \rangle & & & & \end{array}$$

Further develop the following codewords making a short orbit of length 9 and a short orbit of length 6:

$$\langle 0_0, 9_0, 0_1, 9_1 \rangle \quad \langle 0_2, 6_2, 12_2, 0_5 \rangle$$

□

Lemma 4.35 $A_3(7, 6, [3, 1]) = 2$, $A_3(9t + 7, 6, [3, 1]) = U(9t + 7, 6, [3, 1])$ for each $t \in \{1, 2, 3, 6, 24\}$.

Proof: For $t = 0$, see Table 1.

For $t = 1$, the desired code is constructed on I_{16} with 24 codewords listed below.

$$\begin{array}{cccc} \langle 4, 6, 9, 5 \rangle & \langle 9, 0, 3, 13 \rangle & \langle 5, 2, 10, 11 \rangle & \langle 9, 15, 2, 14 \rangle \\ \langle 1, 0, 6, 8 \rangle & \langle 4, 8, 14, 2 \rangle & \langle 15, 6, 12, 7 \rangle & \langle 11, 0, 14, 5 \rangle \\ \langle 8, 5, 7, 6 \rangle & \langle 0, 12, 2, 4 \rangle & \langle 10, 7, 0, 15 \rangle & \langle 8, 13, 15, 0 \rangle \\ \langle 3, 2, 1, 7 \rangle & \langle 3, 5, 15, 4 \rangle & \langle 10, 6, 3, 14 \rangle & \langle 11, 4, 1, 15 \rangle \\ \langle 1, 5, 13, 9 \rangle & \langle 11, 7, 9, 12 \rangle & \langle 14, 7, 13, 3 \rangle & \langle 13, 10, 4, 12 \rangle \\ \langle 9, 10, 8, 1 \rangle & \langle 3, 8, 12, 11 \rangle & \langle 13, 11, 6, 2 \rangle & \langle 1, 12, 14, 10 \rangle \end{array}$$

For $t = 2$, the desired code is constructed on $\mathbb{Z}_{24} \cup \{\infty\}$. Develop the following codewords $+4 \pmod{24}$.

$$\begin{array}{ccccc} \langle 0, 9, 10, 5 \rangle & \langle 21, 5, 18, 9 \rangle & \langle 10, 17, 4, 22 \rangle & \langle 7, 16, 18, 11 \rangle & \langle 11, 17, 0, 12 \rangle \\ \langle 6, 7, 2, 12 \rangle & \langle 9, 8, 12, 11 \rangle & \langle 1, 23, 15, 11 \rangle & \langle 0, 7, 22, 14 \rangle & \langle \infty, 23, 17, 2 \rangle \end{array}$$

Then add two codewords $\langle 0, 8, 16, \infty \rangle$ and $\langle 4, 12, 20, \infty \rangle$.

For $t = 3$, the desired code is constructed on $X = (\mathbb{Z}_6 \times \{0, 1, 2, 3, 4\}) \cup (\mathbb{Z}_2 \times \{5, 6\})$. Define $\alpha : X \rightarrow X$ as $x_y \rightarrow (x + 1)_y$ where the addition is modulo 6 if $y \in \{0, 1, 2, 3, 4\}$, and modulo 2 if $y \in \{5, 6\}$. Develop the following 19 base codewords with α :

$$\begin{array}{cccccc} \langle 0_5, 1_4, 1_1, 0_2 \rangle & \langle 0_5, 3_0, 2_4, 5_2 \rangle & \langle 0_5, 5_3, 2_1, 4_3 \rangle & \langle 0_2, 5_2, 2_3, 0_5 \rangle & \langle 0_6, 3_1, 2_0, 1_3 \rangle & \langle 0_6, 0_3, 2_2, 5_4 \rangle \\ \langle 0_6, 0_4, 5_0, 2_1 \rangle & \langle 3_2, 5_4, 3_3, 0_6 \rangle & \langle 1_2, 0_1, 4_0, 3_0 \rangle & \langle 3_1, 3_3, 3_0, 4_0 \rangle & \langle 5_4, 4_1, 3_1, 2_4 \rangle & \langle 1_0, 5_3, 0_2, 4_0 \rangle \\ \langle 0_1, 2_0, 3_2, 0_2 \rangle & \langle 5_0, 1_3, 2_4, 2_3 \rangle & \langle 4_2, 4_0, 4_4, 5_3 \rangle & \langle 2_2, 0_1, 4_2, 3_4 \rangle & \langle 3_3, 1_4, 0_4, 2_2 \rangle & \langle 4_3, 0_3, 5_1, 2_1 \rangle \\ \langle 0_0, 2_4, 4_4, 5_1 \rangle & & & & & \end{array}$$

Then add the following 5 codewords.

$$\langle 0_0, 2_0, 4_0, 0_5 \rangle \quad \langle 1_0, 3_0, 5_0, 1_5 \rangle \quad \langle 0_5, 1_5, 0_6, 1_6 \rangle \quad \langle 0_1, 2_1, 4_1, 0_6 \rangle \quad \langle 1_1, 3_1, 5_1, 1_6 \rangle$$

For $t = 6$, take a $[3, 1]$ -GDC(6) of type 15^4 from Lemma 4.5. Adjoin one ideal point, and fill in the groups together with the ideal point with optimal codes of length 16 constructed above to obtain the desired code.

Finally, for $t = 24$, take a $[3, 1]$ -GDC(6) of type 18^4 from Lemma 4.6 and assign each point with weight 3 to obtain a $[3, 1]$ -GDC(6) of type 54^4 . Adjoin seven ideal points and fill in the groups together with these ideal points with $[3, 1]$ -GDC(6)s of type $1^{54}7^1$ from Lemma 4.34 to obtain a $[3, 1]$ -GDC(6) of type $1^{162}61^1$. Then fill in the group with an optimal code of length 61 to obtain the desired code. \square

Lemma 4.36 $A_3(9t + 7, 6, [3, 1]) = U(9t + 7, 6, [3, 1])$ for each $t \geq 17$ and $t \notin \{20, 26, 28\}$.

Proof: For each $t \equiv 1, 2, 3 \pmod{4}$, $t \geq 17$ and $t \neq 26$, write $t = 4u + x$ such that $1 \leq x \leq 3$. Take a $[3, 1]$ -GDC(6) of type $36^u(9x)^1$ from Lemmas 4.12 and 4.13. Adjoin seven ideal points, and fill in the groups of size 36 together with these ideal points with $[3, 1]$ -GDC(6)s of type $1^{36}7^1$ from Lemma 4.33 to obtain a $[3, 1]$ -GDC(6) of type $1^{36u}(9x + 7)^1$. Then fill in the group with an optimal code of length $9x + 7$ from Lemma 4.35 to obtain an optimal $(9(4u + x) + 7, 6, [3, 1])_3$ -code, as desired.

Now we consider the case of $t \equiv 0 \pmod{4}$. For $t = 24$, see Lemma 4.35. For $t \geq 120$, write $t = 4u + 24$. Take a $[3, 1]$ -GDC(6) of type 36^u216^1 from Lemma 4.12. Adjoin seven ideal points and fill in the groups. For $32 \leq t < 120$, write $t = 6u - 12 + x + y + z$ with $u \in \{7, 9, 11, 13, 16, 19\}$ such that $x, y, z \in \{0, 2, 4, 6\}$ and at most one of x, y, z takes the value 2. Take a TD(7, u) from Theorem 2.7, and remove one point from one group to obtain a $\{7, u\}$ -GDD of type $6^u(u - 1)^1$. Apply the Fundamental Construction with weight 9 to

the points in $u - 2$ groups of size 6 and weights 0 or 9 to the remaining points. Noting that there exist $[3, 1]$ -GDC(6)s of type 9^s for $s \geq 4$ by Theorem 4.4, we obtain a $[3, 1]$ -GDC(6) of type $54^{u-2}(9x)^1(9y)^1(9z)^1$. Then adjoin seven ideal points and fill in the groups. Note that there exist $[3, 1]$ -GDC(6)s of types $1^{36}7^1$ and $1^{54}7^1$ and optimal codes of lengths 25 and 61. We get an optimal $(9(6u - 12 + x + y + z) + 7, 6, [3, 1])_3$ -code, as desired. \square

Summarizing the above results, we have:

Theorem 4.37 $A_3(7, 6, [3, 1]) = 2$, $A_3(9t + 7, 6, [3, 1]) = U(9t + 7, 6, [3, 1])$ for each $t \geq 1$ and $t \notin \{4, 5\} \cup [7, 16] \cup \{20, 26, 28\}$.

4.7 Case of Length $n \equiv 8 \pmod{9}$

Lemma 4.38 $A_3(8, 6, [3, 1]) = 4$, $A_3(9t + 8, 6, [3, 1]) = U(9t + 8, 6, [3, 1])$ for $t \in \{1, 2\}$.

Proof: For $t = 0$, see Table 1.

For $t = 1$, an optimal $(17, 6, [3, 1])_3$ -code is constructed on I_{17} with 28 codewords listed below.

$$\begin{array}{llllll} \langle 5, 0, 4, 2 \rangle & \langle 0, 3, 16, 9 \rangle & \langle 2, 15, 10, 5 \rangle & \langle 0, 10, 6, 8 \rangle & \langle 12, 5, 6, 15 \rangle & \langle 7, 13, 11, 5 \rangle \\ \langle 7, 6, 9, 3 \rangle & \langle 1, 12, 9, 0 \rangle & \langle 10, 7, 1, 14 \rangle & \langle 11, 8, 4, 3 \rangle & \langle 16, 5, 14, 7 \rangle & \langle 8, 12, 13, 14 \rangle \\ \langle 7, 2, 8, 0 \rangle & \langle 1, 3, 5, 13 \rangle & \langle 7, 4, 15, 12 \rangle & \langle 3, 15, 14, 8 \rangle & \langle 9, 14, 2, 13 \rangle & \langle 15, 11, 9, 16 \rangle \\ \langle 6, 4, 14, 1 \rangle & \langle 1, 2, 11, 6 \rangle & \langle 13, 15, 0, 1 \rangle & \langle 4, 13, 10, 9 \rangle & \langle 1, 16, 8, 15 \rangle & \langle 14, 11, 0, 12 \rangle \\ \langle 5, 8, 9, 10 \rangle & \langle 3, 12, 2, 4 \rangle & \langle 6, 16, 13, 2 \rangle & \langle 16, 10, 12, 11 \rangle & & \end{array}$$

For $t = 2$, the desired code is constructed on $X = (\mathbb{Z}_4 \times \{0, 1, 2, 3, 4, 5\}) \cup (\mathbb{Z}_2 \times \{6\})$. Define $\alpha : X \rightarrow X$ as $x_y \rightarrow (x + 1)_y$ where the addition is modulo 4 if $y \in \{0, 1, 2, 3, 4, 5\}$, and modulo 2 if $y \in \{6\}$. Develop the following 17 base codewords with α :

$$\begin{array}{llllll} \langle 1_3, 2_1, 0_6, 2_5 \rangle & \langle 3_5, 3_4, 0_6, 2_3 \rangle & \langle 0_4, 1_0, 0_6, 3_1 \rangle & \langle 0_0, 2_2, 0_6, 1_2 \rangle & \langle 0_3, 3_1, 3_2, 0_6 \rangle & \langle 0_3, 3_3, 1_2, 1_0 \rangle \\ \langle 1_5, 3_2, 3_3, 0_4 \rangle & \langle 2_4, 2_3, 1_5, 0_1 \rangle & \langle 3_5, 0_5, 0_2, 3_0 \rangle & \langle 3_4, 3_1, 0_1, 1_4 \rangle & \langle 2_0, 2_1, 2_3, 0_3 \rangle & \langle 1_0, 2_3, 1_4, 2_5 \rangle \\ \langle 0_4, 1_4, 1_2, 3_2 \rangle & \langle 3_2, 2_1, 0_5, 0_1 \rangle & \langle 2_1, 3_0, 1_5, 3_5 \rangle & \langle 0_0, 1_1, 3_2, 0_2 \rangle & \langle 0_5, 1_0, 2_4, 0_3 \rangle & \end{array}$$

Further add the codeword $\langle 0_0, 1_0, 2_0, 3_0 \rangle$. \square

Lemma 4.39 There is a $[3, 1]$ -GDC(6) of type $1^{9t}8^1$ with size $9t^2 + 14t$ for each $t \in \{3, 4, 5, 6\}$.

Proof: For each t , the desired GDC is constructed on $X = (\mathbb{Z}_{3t} \times \{0, 1, 2\}) \cup (\mathbb{Z}_3 \times \{3, 4\}) \cup \{\infty_0, \infty_1\}$, where the point set $(\mathbb{Z}_3 \times \{3, 4\}) \cup \{\infty_0, \infty_1\}$ forms the group of size 8. Define $\alpha : X \rightarrow X$ as $x_y \rightarrow (x + 1)_y$ where the addition is modulo $3t$ if $y \in \{0, 1, 2\}$, and modulo 3 if $y \in \{3, 4\}$. The points ∞_0, ∞_1 are fixed by α . First develop the following two codewords with α , making two short orbits of length t :

$$\langle 0_0, t_0, (2t)_0, \infty_0 \rangle \quad \langle 0_1, t_1, (2t)_1, \infty_1 \rangle$$

Then develop the following base codewords with α :

$t = 3$:

$$\begin{array}{llllll} \langle 0_3, 0_1, 8_1, 7_0 \rangle & \langle 0_3, 0_0, 8_0, 1_2 \rangle & \langle 0_3, 0_2, 4_1, 5_2 \rangle & \langle 8_2, 4_0, 7_2, 0_3 \rangle & \langle 0_4, 2_2, 0_1, 1_0 \rangle & \langle 0_4, 6_2, 4_2, 5_1 \rangle \\ \langle 0_4, 0_0, 2_0, 1_1 \rangle & \langle 1_0, 7_1, 5_1, 0_4 \rangle & \langle 0_0, 0_2, 4_0, 0_1 \rangle & \langle \infty_0, 6_1, 4_2, 6_0 \rangle & \langle 6_1, 8_0, 2_1, 6_2 \rangle & \langle 3_1, 0_2, 6_2, 7_0 \rangle \\ \langle \infty_1, 6_2, 0_0, 2_1 \rangle & & & & & \end{array}$$

$t = 4$:

$$\begin{array}{llllll} \langle \infty_0, 4_1, 11_2, 7_0 \rangle & \langle 5_1, 1_2, 6_2, 7_2 \rangle & \langle 0_3, 3_1, 10_1, 8_2 \rangle & \langle 3_2, 5_2, 8_0, 0_3 \rangle & \langle 0_0, 0_1, 1_0, 7_0 \rangle \\ \langle 2_1, 11_1, 0_1, 11_2 \rangle & \langle 0_4, 0_2, 8_2, 6_1 \rangle & \langle 5_0, 3_2, 0_1, 10_0 \rangle & \langle 0_4, 0_0, 10_1, 4_2 \rangle & \langle 1_2, 9_1, 5_0, 0_4 \rangle \\ \langle 0_4, 10_0, 11_1, 8_0 \rangle & \langle 0_3, 2_1, 6_0, 8_0 \rangle & \langle 7_2, 5_0, 8_0, 2_1 \rangle & \langle 0_3, 4_2, 10_0, 3_2 \rangle & \langle 3_0, 3_2, 6_2, 6_1 \rangle \\ \langle \infty_1, 9_2, 8_0, 10_1 \rangle & & & & \end{array}$$

$t = 5$:

$$\begin{array}{llllll} \langle 9_1, 5_2, 1_0, 3_2 \rangle & \langle 0_1, 3_1, 7_1, 4_2 \rangle & \langle 0_4, 2_2, 12_1, 13_1 \rangle & \langle 11_0, 6_1, 5_0, 8_0 \rangle & \langle 0_3, 11_0, 2_2, 0_0 \rangle \\ \langle 7_1, 0_0, 9_1, 0_3 \rangle & \langle 0_3, 2_1, 9_2, 1_1 \rangle & \langle 5_1, 14_0, 11_1, 6_0 \rangle & \langle 0_4, 12_2, 5_0, 1_0 \rangle & \langle 5_2, 8_2, 10_1, 10_2 \rangle \\ \langle 0_4, 2_1, 0_0, 1_2 \rangle & \langle 4_1, 4_0, 7_2, 0_4 \rangle & \langle 2_1, 13_0, 14_0, 10_2 \rangle & \langle 11_0, 1_2, 5_2, 4_0 \rangle & \langle \infty_0, 2_1, 4_2, 6_0 \rangle \\ \langle 0_2, 1_2, 7_2, 1_1 \rangle & \langle 0_3, 7_2, 7_0, 6_1 \rangle & \langle \infty_1, 11_0, 10_2, 1_1 \rangle & \langle 12_2, 2_0, 4_0, 0_1 \rangle & \end{array}$$

$t = 6$:

$$\begin{array}{llllll} \langle 17_0, 14_1, 17_1, 13_2 \rangle & \langle 0_0, 2_2, 12_1, 9_2 \rangle & \langle 14_1, 1_0, 7_1, 12_2 \rangle & \langle 3_0, 16_0, 14_0, 0_3 \rangle & \langle 0_4, 4_1, 0_1, 13_0 \rangle \\ \langle 16_2, 10_0, 2_2, 13_2 \rangle & \langle 0_4, 0_0, 15_2, 8_1 \rangle & \langle 12_1, 4_2, 12_2, 1_0 \rangle & \langle 12_1, 13_0, 2_1, 4_0 \rangle & \langle 14_1, 2_2, 4_0, 0_4 \rangle \\ \langle 5_2, 11_2, 10_0, 14_2 \rangle & \langle 17_2, 6_1, 0_2, 1_1 \rangle & \langle 0_4, 13_2, 14_0, 8_2 \rangle & \langle 0_3, 0_1, 17_1, 15_0 \rangle & \langle 0_0, 0_2, 10_0, 3_1 \rangle \\ \langle 15_0, 16_0, 12_0, 17_1 \rangle & \langle 0_3, 6_2, 4_2, 10_0 \rangle & \langle \infty_1, 12_2, 5_0, 3_1 \rangle & \langle 0_1, 2_2, 16_1, 7_2 \rangle & \langle 0_3, 17_2, 4_1, 8_0 \rangle \\ \langle \infty_0, 15_1, 0_2, 14_0 \rangle & \langle 1_1, 2_2, 15_0, 6_1 \rangle & & & \end{array}$$

□

Lemma 4.40 $A_3(9t + 8, 6, [3, 1]) = U(9t + 8, 6, [3, 1])$ for each $t \geq 13$ and $t \notin \{15, 23, 28\}$.

Proof: For each $t \in \{13, 16, 19\}$, take a $[3, 1]$ -GDC(6) of type $27^s 9^1$ with $s \in \{4, 5, 6\}$ from Lemma 4.9 and adjoin eight ideal points. Then fill in the groups together with the ideal points with $[3, 1]$ -GDC(6)s of type $1^{27} 8^1$ and an optimal code of length 17 to obtain the desired code. For each $t \in \{14, 20\}$, take a $[3, 1]$ -GDC(6) of type $27^s 18^1$ with $s \in \{4, 6\}$ from Lemma 4.10 and adjoin eight ideal points. Then fill in the groups with $[3, 1]$ -GDC(6)s of type $1^{27} 8^1$ and an optimal code of length 26 to obtain the desired code. For $t \in \{17, 18\}$, take a $[3, 1]$ -GDC(6) of type $36^4 9^1$ or type $36^4 18^1$ from Lemma 4.12, adjoin eight ideal points and fill in the groups.

For $t \in \{21, 22, 24, 25, 26, 27\}$, take a TD(6, 5) from Theorem 2.7 and apply the Fundamental Construction to this TD, assigning weight 9 to the points in the first four groups and weights 0 or 9 to the remaining points. Thus we can get a $[3, 1]$ -GDC(6) of type $45^4 (9x)^1 (9y)^1$

with $x \in \{0, 3, 4, 5\}$ and $y \in \{1, 2\}$. Then adjoin eight ideal points and fill in the groups. For $t \in \{29, 30\}$, take a $[3, 1]$ -GDC(6) of type $36^7 9^1$ or type $36^7 18^1$ from Lemma 4.12, adjoin eight ideal points and fill in the groups.

For $31 \leq t \leq 80$ and $t \notin \{33, 57\}$, take a $\text{TD}(7, u)$ with $u \in \{7, 8, 9, 11, 12, 13\}$ from Theorem 2.7, and remove one point from one group to obtain a $\{7, u\}$ -GDD of type $6^u(u-1)^1$. Apply the Fundamental Construction to this GDD, assigning weight 9 to the points in the first $u-2$ groups of size 6 and weights 0 or 9 to the remaining points. We get a $[3, 1]$ -GDC(6) of type $54^{u-2}(9x)^1(9y)^1(9z)^1$ with $x, y \in \{0, 3, 4, 5, 6\}$ and $z \in \{1, 2\}$. Then adjoin eight ideal points and fill in the groups. For $t \in \{33, 57\}$, take a $[3, 1]$ -GDC(6) of type $36^8 9^1$ or type $36^{14} 9^1$ from Lemma 4.12, adjoin eight ideal points and fill in the groups.

Finally, for $t \geq 80$ and $t \notin \{83, 87, 91\}$, take a $[3, 1]$ -GDC(6) of type $36^u(9x)^1$ with $u \geq 16$ and $x \in \{1, 2, 16\}$ or $u \geq 19$ and $x = 19$ from Lemma 4.12. Then adjoin eight ideal points and fill in the groups. For each $t \in \{83, 87, 91\}$, take a $\text{TD}(6, u)$ with $u = 15, 17$ or 19 from Theorem 2.7, and remove one point from one group to obtain a $\{6, u\}$ -GDD of type $5^u(u-1)^1$. Apply the Fundamental Construction to this GDD, assigning weight 9 to the points in the first $u-1$ groups of size 5 and weights 0 or 9 to the remaining points. Thus we can get a $[3, 1]$ -GDC(6) of type $45^{14} 117^1$, type $45^{17} 18^1$ or type $45^{18} 9^1$. Then adjoin eight ideal points and fill in the groups to complete the proof. \square

Summarizing the above results, we have:

Theorem 4.41 $A_3(8, 6, [3, 1]) = 4$, $A_3(9t + 8, 6, [3, 1]) = U(9t + 8, 6, [3, 1])$ for each $t \geq 1$ and $t \notin [3, 12] \cup \{15, 23, 28\}$.

5 Conclusion

In this paper, we determine almost completely the spectrum of sizes for optimal ternary constant-composition codes with weight four and minimum distance six. We summarize our main results of this paper as follows:

Theorem 5.1 For any integer $n \geq 4$, let $Q = \{13, 16, 22, 59, 65, 71, 76, 88, 94, 124\}$, $Q_1 = \{14, 23, 29, 35, 41, 47, 53, 83\} \cup \{n : 95 \leq n \leq 323, n \equiv 11, 17, 23 \pmod{24}\} \cup \{347, 353, 359, 371, 377\}$, $Q_2 = \{17, 89\}$. Then

$$A_3(n, 6, [2, 2]) = \begin{cases} 1, & \text{if } n \in \{4, 5\}; \\ 3, & \text{if } n = 7; \\ 5, & \text{if } n = 8; \\ 15, & \text{if } n = 11; \\ \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{3} \rfloor, & \text{if } n \geq 6 \text{ and } n \notin \{7, 8, 11\} \cup Q \cup Q_1 \cup Q_2. \end{cases}$$

Furthermore, we have

1. $\lfloor \frac{n}{2} \lfloor \frac{n-1}{3} \rfloor \rfloor - 1 \leq A_3(n, 6, [2, 2]) \leq \lfloor \frac{n}{2} \lfloor \frac{n-1}{3} \rfloor \rfloor$ when $n \in Q_1$;
2. $\lfloor \frac{n}{2} \lfloor \frac{n-1}{3} \rfloor \rfloor - 2 \leq A_3(n, 6, [2, 2]) \leq \lfloor \frac{n}{2} \lfloor \frac{n-1}{3} \rfloor \rfloor$ when $n \in Q_2$.

Theorem 5.2 *For any integer $n \geq 4$, write $n = 9t + i$ with $0 \leq i < 9$. Then we have:*

1. $i = 0$: $A_3(9t, 6, [3, 1]) = 9t^2 - 3t$ for each $t \geq 1$;
2. $i = 1$: $A_3(9t + 1, 6, [3, 1]) = 9t^2 + t$ for each $t \geq 1$, except possibly for $t = 2$;
3. $i = 2$: $A_3(9t + 2, 6, [3, 1]) = 9t^2 + 2t$ for each $t \geq 1$;
4. $i = 3$: $A_3(9t + 3, 6, [3, 1]) = 9t^2 + 3t$ for each $t \geq 1$;
5. $i = 4$: $A_3(9t + 4, 6, [3, 1]) = 9t^2 + 6t + 1 + \lfloor \frac{t}{4} \rfloor$ for each $t \geq 0$ and $t \notin \{1, 5, 6, 7, 9, 10, 11, 13, 14, 15, 21, 25, 26, 29, 37, 41, 45, 49, 53, 57, 61, 65, 69, 73, 77, 81\}$;
6. $i = 5$: $A_3(9t + 5, 6, [3, 1]) = 9t^2 + 7t + 1 + \lfloor \frac{t+1}{4} \rfloor$ for each $t \geq 0$ and $t \notin [5, 15] \cup \{26\}$;
7. $i = 6$: $A_3(9t + 6, 6, [3, 1]) = 9t^2 + 9t + 2$ for each $t \geq 0$;
8. $i = 7$: $A_3(9t + 7, 6, [3, 1]) = 9t^2 + 11t + 3 + \lfloor \frac{t+1}{2} \rfloor$ for each $t \geq 1$ and $t \notin \{4, 5\} \cup [7, 16] \cup \{20, 26, 28\}$;
9. $i = 8$: $A_3(9t + 8, 6, [3, 1]) = 9t^2 + 14t + 5$ for each $t \geq 1$ and $t \notin [3, 12] \cup \{15, 23, 28\}$.

Acknowledgements

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